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THE

MESSENGER OF MATHEMATICS.

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MESSENGER OF MATHEMATICS.

ON THE SERIES WHICH REPRESENT THE TWELVE ELLIPTIC AND THE FOUR ZETA FUNCTIONS.

By *J. W. L. Glaisher.*

Introduction, § 1.

§ 1. In this paper I propose to consider the different systems of series by which the elliptic and Zeta functions are represented. These functions are of so fundamental a character in Analysis, and possess such remarkable and distinctive properties, that it is interesting to examine in some detail the form and structure of the series which express them.

The sixteen functions form one system. It is true that the Zeta functions are not doubly periodic, like the elliptic functions, but they are very nearly so; and, so to speak, they only just miss the property of complete double periodicity. An inspection of any of the systems of series for the sixteen functions shews not only that there exists a resemblance between the series representing the elliptic functions and those representing the Zeta functions, but that the latter are actually required in order to complete the four groups into which the former separate themselves. In fact, so perfectly alike in appearance are the representations of the elliptic and Zeta functions that it is always a curious and interesting matter to investigate, in the case of each system of series, the manner in which it is brought about that the Zeta series fail to attain the same amount of periodicity as that which is enjoyed by the others.

This paper is not intended as a continuation of one which appeared with a nearly similar title* at the beginning of the last volume of the *Messenger*, but it may be regarded as forming a sequel to some portions of that paper.

* "On the transformation and developments of the twelve elliptic functions four Zeta functions," vol. XVII., pp. 1–18.

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The notation $\mathrm{zs}\,x$, &c., § 2.

§ 2. The function $Z(x)$ is specially related to the group $\mathrm{sn}\,x$, $\mathrm{cn}\,x$, $\mathrm{dn}\,x$, and the functions $Z_1(x)$, $Z_2(x)$, $Z_3(x)$ are similarly related to the other three groups $\mathrm{ns}\,x$, $\mathrm{ds}\,x$, $\mathrm{cs}\,x$; $\mathrm{dc}\,x$, $\mathrm{nc}\,x$, $\mathrm{sc}\,x$; $\mathrm{cd}\,x$, $\mathrm{sd}\,x$, $\mathrm{nd}\,x$ respectively. It thus appears that a two-letter notation for the four Zeta functions, in which they are distinguished from each other by the four final letters n, s, c, d would be more representative than one in which the distinction is made by means of the suffixes 0, 1, 2, 3. I therefore denote the four Zeta functions by $\mathrm{zn}\,x$, $\mathrm{zs}\,x$, $\mathrm{zc}\,x$, $\mathrm{zd}\,x$, so that

$$\mathrm{zn}\,x = Z(x), \quad \mathrm{zs}\,x = Z_1(x), \quad \mathrm{zc}\,x = Z_2(x), \quad \mathrm{zd}\,x = Z_3(x).$$

For several reasons it is desirable, in writing the system of sixteen functions, to place the s-group first instead of the n-group.

The q-series for the sixteen functions have been given in their ordinary form in § 2 (pp. 2, 3) of the paper in the last volume. In order to express them as series proceeding by powers of q the following notation and definitions are introduced.

The notation d, d', δ, δ', § 3.

§ 3. Let n denote any number. Let d denote any divisor of n, and let d' be its conjugate.* Also let δ denote any uneven divisor of n, and let δ' be its conjugate. If n be uneven, all its divisors are uneven, so that in this case d and d' are identical with δ and δ' respectively.

The letter m will be used to denote any uneven number.

The symbols σ, ζ, ζ', &c., §§ 4, 5.

§ 4. Let $\sigma\phi(n)$ denote

$$\phi(d_1) + \phi(d_2) \ldots + \phi(d_r),$$

$d_1, d_2, \ldots, d_r$ being all the divisors of n; or, as we may express it, let

$$\sigma\phi(n) = \Sigma_n \phi(d),$$

the suffix n attached to the Σ denoting the number to which the divisors have reference.

* If d be any divisor of n, its conjugate is $\frac{n}{d}$. Thus $dd' = \delta\delta' = n$.

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Similarly, let

$$\zeta\phi(n) = \Sigma_n(-)^{d-1}\phi(d),$$

$$\zeta'\phi(n) = \Sigma_n(-)^{d'-1}\phi(d),$$

and

$$\xi\phi(n) = \Sigma_n(-)^{d+d'}\phi(d).$$

In the formation of the expression $\zeta\phi(n)$, therefore, the terms in which the argument is uneven are to have the positive sign, and the terms in which it is even are to have the negative sign. In the expression $\zeta'\phi(n)$ the positive terms are those in which the conjugate of the argument is uneven, and the negative terms are those in which the conjugate is even.

In the expression $\xi\phi(n)$ the positive terms are those in which the sum of the argument and its conjugate is even, and the negative terms are those in which the sum is uneven.

When n is uneven the four functions σ, ζ, ζ', ξ become identical, and all the terms are positive. When n is even the positive terms are those having respectively as arguments (i) all the divisors, (ii) the uneven divisors, (iii) the conjugates of the uneven divisors, (iv) the even divisors whose conjugates are also even.

Also, let

$$\Delta\phi(n) = \Sigma_n\phi(\delta),$$

$$\Delta'\phi(n) = \Sigma_n\phi(\delta'),$$

and

$$E\phi(n) = \Sigma_n(-)^{\frac{1}{2}(\delta-1)}\phi(\delta),$$

$$E'\phi(n) = \Sigma_n(-)^{\frac{1}{2}(\delta-1)}\phi(\delta').$$

Thus in the case of $\Delta\phi(n)$ and $E\phi(n)$ the arguments are the uneven divisors of n. In $\Delta\phi(n)$ all the terms have the positive sign; in $E\phi(n)$ the terms in which the argument is of the form $4r+1$ are positive, and those in which it is of the form $4r+3$ are negative.

In $\Delta'\phi(n)$ and $E'\phi(n)$ the arguments are those divisors of n which have uneven conjugates. In $\Delta'\phi(n)$ all the terms have the positive sign; in $E'\phi(n)$ the terms are positive when the conjugate is of the form $4r+1$ and negative when it is of the form $4r+3$.

In all cases unity and n are included among the divisors of n.

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When n is uneven, all the functions except E and E' become equal to each other, viz.

$$\sigma\phi(m) = \Delta\phi(m) = \Delta'\phi(m) = \zeta\phi(m) = \zeta'\phi(m) = \xi\phi(m) = \Sigma_m\phi(\delta),$$

and

$$E\phi(m) = E'\phi(m) = \Sigma_m(-)^{\frac{1}{2}(\delta-1)}\phi(\delta).$$

§ 5. It may be remarked that ξ, ζ, ζ', Δ, Δ' are connected by the relations

$$\begin{aligned} -\xi(x) &= \zeta\phi(n) + 2(-)^n\Delta'(n) \\ &= \zeta'\phi(n) + 2(-)^n\Delta(n). \end{aligned}$$

System of sixteen algebraical series and their expanded values, § 6.

§ 6. Consider the system of series whose general term is of the form

$$\frac{\phi(r)\,x^r}{1 \pm x^s},$$

and in which (i) s is equal either to r or to $2r$, (ii) r denotes either any number or any uneven number, (iii) the signs in the denominators are either all positive or all negative, (iv) the signs of the terms are either all positive or alternately positive and negative.

The system therefore consists of sixteen series, and it is found that the coefficients in the expanded values of these series in powers of x depend upon the functions σ, ζ, ζ', &c., which have been defined in § 4, viz. denoting any number by n and any uneven number by m, we have:

$$\Sigma_1^\infty \frac{\phi(n)\,x^n}{1-x^n} = \Sigma_1^\infty \sigma\phi(n)\,x^n = \Sigma_1^\infty \Sigma_n \phi(d)\,x^n,$$

$$\Sigma_1^\infty (-)^{n-1}\frac{\phi(n)\,x^n}{1-x^n} = \Sigma_1^\infty \zeta\phi(n)\,x^n = \Sigma_1^\infty \Sigma_n (-)^{d-1}\phi(d)\,x^n,$$

$$\Sigma_1^\infty \frac{\phi(n)\,x^n}{1+x^n} = \Sigma_1^\infty \zeta'\phi(n)\,x^n = \Sigma_1^\infty \Sigma_n (-)^{d'-1}\phi(d)\,x^n,$$

$$\Sigma_1^\infty (-)^{n-1}\frac{\phi(n)\,x^n}{1+x^n} = \Sigma_1^\infty \xi\phi(n)\,x^n = \Sigma_1^\infty \Sigma_n (-)^{d+d'}\phi(d)\,x^n;$$

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$$\Sigma_1^\infty \frac{\phi(n)\,x^n}{1-x^{2n}} = \Sigma_1^\infty \Delta'\phi(n)\,x^n = \Sigma_1^\infty \Sigma_n \phi(\delta')\,x^n,$$

$$\Sigma_1^\infty (-)^{n-1}\frac{\phi(n)\,x^n}{1-x^{2n}} = \Sigma_1^\infty (-)^{n-1}\Delta'\phi(n)\,x^n = \Sigma_1^\infty \Sigma_n (-)^{n-1}\phi(\delta')\,x^n,$$

$$\Sigma_1^\infty \frac{\phi(n)\,x^n}{1+x^{2n}} = \Sigma_1^\infty E'\phi(n)\,x^n = \Sigma_1^\infty \Sigma_n (-)^{\frac{1}{2}(\delta-1)}\phi(\delta')\,x^n,$$

$$\Sigma_1^\infty (-)^{n-1}\frac{\phi(n)\,x^n}{1+x^{2n}} = \Sigma_1^\infty (-)^{n-1}E'\phi(n)\,x^n = \Sigma_1^\infty \Sigma_n (-)^{n-1+\frac{1}{2}(\delta-1)}\phi(\delta')\,x^n;$$

$$\Sigma_1^\infty \frac{\phi(m)x^m}{1-x^m} = \Sigma_1^\infty \Delta\phi(n)x^n = \Sigma_1^\infty \Sigma_n \phi(\delta)\,x^n,$$

$$\Sigma_1^\infty (-)^{\frac{1}{2}(m-1)}\frac{\phi(m)x^m}{1-x^m} = \Sigma_1^\infty E\phi(n)x^n = \Sigma_1^\infty \Sigma_n (-)^{\frac{1}{2}(\delta-1)}\phi(\delta)\,x^n,$$

$$\Sigma_1^\infty \frac{\phi(m)x^m}{1+x^m} = \Sigma_1^\infty (-)^{n-1}\Delta\phi(n)x^n = \Sigma_1^\infty \Sigma_n (-)^{n-1}\phi(\delta)\,x^n,$$

$$\Sigma_1^\infty (-)^{\frac{1}{2}(m-1)}\frac{\phi(m)x^m}{1+x^m} = \Sigma_1^\infty (-)^{n-1}E\phi(n)x^n = \Sigma_1^\infty \Sigma_n (-)^{n-1+\frac{1}{2}(\delta-1)}\phi(\delta)x^n;$$

$$\Sigma_1^\infty \frac{\phi(m)\,x^m}{1-x^{2m}} = \Sigma_1^\infty \Delta\phi(m)\,x^m = \Sigma_1^\infty \Sigma_m \phi(\delta)\,x^m,$$

$$\Sigma_1^\infty (-)^{\frac{1}{2}(m-1)}\frac{\phi(m)\,x^m}{1-x^{2m}} = \Sigma_1^\infty E\phi(m)\,x^m = \Sigma_1^\infty \Sigma_m (-)^{\frac{1}{2}(\delta-1)}\phi(\delta)\,x^m,$$

$$\Sigma_1^\infty \frac{\phi(m)\,x^m}{1+x^{2m}} = \Sigma_1^\infty (-)^{\frac{1}{2}(m-1)}E\phi(m)\,x^m = \Sigma_1^\infty \Sigma_m (-)^{\frac{1}{2}(m-1)+\frac{1}{2}(\delta-1)}\phi(\delta)\,x^m,$$

$$\Sigma_1^\infty (-)^{\frac{1}{2}(m-1)}\frac{\phi(m)x^m}{1+x^{2m}} = \Sigma_1^\infty (-)^{\frac{1}{2}(m-1)}\Delta\phi(m)x^m = \Sigma_1^\infty \Sigma_m (-)^{\frac{1}{2}(m-1)}\phi(\delta)\,x^m.$$

Series proceeding by ascending powers of q for the sixteen elliptic and Zeta functions, §§ 7–14.

§ 7. By comparing the q-series given in pp. 2, 3, vol. XVII. for the sixteen elliptic and Zeta functions with the sixteen series considered in the last section, it will be seen that the two systems become identical if in the latter system we assign

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to x the values $q^{\frac{1}{2}}$, q, or $q^{\frac{3}{2}}$, and to $\phi(n)$ and $\phi(m)$ the values $\cos 2nx$ or $\sin 2nx$ and $\cos nx$ or $\sin mx$ respectively.

The elliptic and Zeta functions are in fact the representatives of the system of series in the last section, in certain special cases in which ϕ is either a sine or a cosine. In ten of the series it is a sine and in the other six a cosine.

The following are the values of the sixteen elliptic and Zeta functions expressed as series proceeding by powers of q. The letter ρ denotes $\frac{2K}{\pi}$ and M denotes $\frac{1}{2}(m-1)$.

$$\rho\,\mathrm{ns}\,\rho x = \operatorname{cec} x + 4\Sigma_1^\infty \Delta(\sin nx)\, q^n,$$

$$\rho\,\mathrm{ds}\,\rho x = \operatorname{cec} x + 4\Sigma_1^\infty (-)^n \Delta(\sin nx)\, q^n,$$

$$\rho\,\mathrm{cs}\,\rho x = \cot x - 4\Sigma_1^\infty \zeta'(\sin 2nx)\, q^{2n},$$

$$\rho\,\mathrm{zs}\,\rho x = \cot x + 4\Sigma_1^\infty \sigma(\sin 2nx)\, q^{2n};$$

$$\rho\,\mathrm{dc}\,\rho x = \sec x + 4\Sigma_1^\infty E(\cos nx)\, q^n,$$

$$k'\rho\,\mathrm{nc}\,\rho x = \sec x + 4\Sigma_1^\infty (-)^n E(\cos nx)\, q^n,$$

$$k'\rho\,\mathrm{sc}\,\rho x = \tan x - 4\Sigma_1^\infty \xi(\sin 2nx)\, q^{2n},$$

$$\rho\,\mathrm{zc}\,\rho x = -\tan x - 4\Sigma_1^\infty \zeta(\sin 2nx)\, q^{2n};$$

$$k\rho\,\mathrm{cd}\,\rho x = 4\Sigma_1^\infty E(\cos mx)\, q^{\frac{1}{2}m},$$

$$kk'\rho\,\mathrm{sd}\,\rho x = 4\Sigma_1^\infty (-)^M \Delta(\sin mx)\, q^{\frac{1}{2}m},$$

$$k'\rho\,\mathrm{nd}\,\rho x = 1 + 4\Sigma_1^\infty (-)^n E'(\cos 2nx)\, q^n,$$

$$\rho\,\mathrm{zd}\,\rho x = 4\Sigma_1^\infty (-)^n \Delta'(\sin 2nx)\, q^n;$$

$$k\rho\,\mathrm{sn}\,\rho x = 4\Sigma_1^\infty \Delta(\sin mx)\, q^{\frac{1}{2}m},$$

$$k\rho\,\mathrm{cn}\,\rho x = 4\Sigma_1^\infty (-)^M E(\cos mx)\, q^{\frac{1}{2}m},$$

$$\rho\,\mathrm{dn}\,\rho x = 1 + 4\Sigma_1^\infty E'(\cos 2nx)\, q^n,$$

$$\rho\,\mathrm{zn}\,\rho x = 4\Sigma_1^\infty \Delta'(\sin 2nx)\, q^n.$$

It is to be noticed that all the symbols σ, ζ, ζ', &c. are supposed to have reference to the letters n or m only. When therefore the argument is $2nx$, it is not with the divisors of $2n$ that we are concerned, but with those of n; for example, $\sigma(\sin 2nx)$ denotes

$$\sin 2d_1x + \sin 2d_2x + \ldots + \sin 2d_rx,$$

where $d_1, d_2, \ldots, d_r$ are the divisors of n.

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§ 8. For convenience of reference the expanded form of each series as far as the fourth term involving powers of q is given below.

$$\rho\,\mathrm{ns}\,\rho x = \mathrm{cec}\,x + 4\,\{q\sin x + q^2\sin x + q^3(\sin x + \sin 3x) + q^4\sin x + \&c.\},$$

$$\rho\,\mathrm{ds}\,\rho x = \mathrm{cec}\,x - 4\,\{q\sin x - q^2\sin x + q^3(\sin x + \sin 3x) - q^4\sin x + \&c.\},$$

$$\rho\,\mathrm{cs}\,\rho x = \cot x - 4\,\{q^2\sin 2x - q^4(\sin 2x - \sin 4x) + q^6(\sin 2x + \sin 6x) - q^8(\sin 2x + \sin 4x - \sin 8x) + \&c.\},$$

$$\rho\,\mathrm{zs}\,\rho x = \cot x + 4\,\{q^2\sin 2x + q^4(\sin 2x + \sin 4x) + q^6(\sin 2x + \sin 6x) + q^8(\sin 2x + \sin 4x + \sin 8x) + \&c.\};$$

$$\rho\,\mathrm{dc}\,\rho x = \sec x + 4\,\{q\cos x + q^2\cos x + q^3(\cos x - \cos 3x) + q^4\cos x + \&c.\},$$

$$k'\rho\,\mathrm{nc}\,\rho x = \sec x - 4\,\{q\cos x - q^2\cos x + q^3(\cos x - \cos 3x) - q^4\cos x + \&c.\},$$

$$k'\rho\,\mathrm{sc}\,\rho x = \tan x - 4\,\{q^2\sin 2x - q^4(\sin 2x + \sin 4x) + q^6(\sin 2x + \sin 6x) - q^8(\sin 2x - \sin 4x - \sin 8x)\},$$

$$\rho\,\mathrm{zc}\,\rho x = -\tan x - 4\,\{q^2\sin 2x + q^4(\sin 2x - \sin 4x) + q^6(\sin 2x + \sin 6x) + q^8(\sin 2x - \sin 4x - \sin 8x) + \&c.\};$$

$$k\rho\,\mathrm{cd}\,\rho x = 4\,\{q^{\frac{1}{2}}\cos x + q^{\frac{3}{2}}(\cos x - \cos 3x) + q^{\frac{5}{2}}(\cos x + \cos 5x) + q^{\frac{7}{2}}(\cos x - \cos 7x) + \&c.\},$$

$$kk'\rho\,\mathrm{sd}\,\rho x = 4\,\{q^{\frac{1}{2}}\sin x - q^{\frac{3}{2}}(\sin x + \sin 3x) + q^{\frac{5}{2}}(\sin x + \sin 5x) - q^{\frac{7}{2}}(\sin x + \sin 7x) + \&c.\},$$

$$k'\rho\,\mathrm{nd}\,\rho x = 1 - 4\,\{q\cos 2x - q^2\cos 4x - q^3(\cos 2x - \cos 6x) - q^4\cos 8x + \&c.\},$$

$$\rho\,\mathrm{zd}\,\rho x = -4\,\{q\sin 2x - q^2\sin 4x + q^3(\sin 2x + \sin 6x) - q^4\sin 8x + \&c.\};$$

$$k\rho\,\mathrm{sn}\,\rho x = 4\,\{q^{\frac{1}{2}}\sin x + q^{\frac{3}{2}}(\sin x + \sin 3x) + q^{\frac{5}{2}}(\sin x + \sin 5x) + q^{\frac{7}{2}}(\sin x + \sin 7x) + \&c.\},$$

$$k\rho\,\mathrm{cn}\,\rho x = 4\,\{q^{\frac{1}{2}}\cos x - q^{\frac{3}{2}}(\cos x - \cos 3x) + q^{\frac{5}{2}}(\cos x + \cos 5x) - q^{\frac{7}{2}}(\cos x - \cos 7x) + \&c.\},$$

$$\rho\,\mathrm{dn}\,\rho x = 1 + 4\,\{q\cos 2x + q^2\cos 4x - q^3(\cos 2x - \cos 6x) + q^4\cos 8x + \&c.\},$$

$$\rho\,\mathrm{zn}\,\rho x = 4\,\{q\sin 2x + q^2\sin 4x + q^3(\sin 2x + \sin 6x) + q^4\sin 8x + \&c.\}.$$

§ 9. If the sixteen functions be arranged in groups depending upon similarity of form in their q-series, they separate themselves into groups of four, which are not the same as the groups of four into which the functions have been divided in the last two sections, *i.e.* in the new grouping

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the second letter of the functional sign is not the same for each group. This is evident from the q-series themselves in their original form (pp. 2, 3, vol. XVII), or from their expanded forms in § 7.

This new grouping of the formulæ is shown below, the system given in § 7 being divided into groups depending upon the form of the q-series.

$$k\rho\,\mathrm{sn}\,\rho x = 4\Sigma_1^\infty \quad \Delta(\sin mx)\,q^{\frac{1}{2}m},$$
$$kk'\rho\,\mathrm{sd}\,\rho x = 4\Sigma_1^\infty (-)^M \Delta(\sin mx)\,q^{\frac{1}{2}m},$$
$$k\rho\,\mathrm{cd}\,\rho x = 4\Sigma_1^\infty \quad E(\cos mx)\,q^{\frac{1}{2}m},$$
$$k\rho\,\mathrm{cn}\,\rho x = 4\Sigma_1^\infty (-)^M E(\cos mx)\,q^{\frac{1}{2}m};$$

$$\rho\,\mathrm{zn}\,\rho x = \quad 4\Sigma_1^\infty \quad \Delta'(\sin 2nx)\,q^n,$$
$$\rho\,\mathrm{zd}\,\rho x = \quad 4\Sigma_1^\infty (-)^n \Delta'(\sin 2nx)\,q^n,$$
$$\rho\,\mathrm{dn}\,\rho x = 1 + 4\Sigma_1^\infty \quad E'(\cos 2nx)\,q^n,$$
$$k'\rho\,\mathrm{nd}\,\rho x = 1 + 4\Sigma_1^\infty (-)^n E'(\cos 2nx)\,q^n;$$

$$\rho\,\mathrm{ns}\,\rho x = \mathrm{cec}\,x + 4\Sigma_1^\infty \quad \Delta(\sin nx)\,q^n,$$
$$\rho\,\mathrm{ds}\,\rho x = \mathrm{cec}\,x + 4\Sigma_1^\infty (-)^n \Delta(\sin nx)\,q^n,$$
$$\rho\,\mathrm{dc}\,\rho x = \sec x + 4\Sigma_1^\infty \quad E(\cos nx)\,q^n,$$
$$k'\rho\,\mathrm{nc}\,\rho x = \sec x + 4\Sigma_1^\infty (-)^n E(\cos nx)\,q^n;$$

$$\rho\,\mathrm{zs}\,\rho x = \quad \cot x + 4\Sigma_1^\infty \sigma(\sin 2nx)\,q^{2n},$$
$$\rho\,\mathrm{cs}\,\rho x = \quad \cot x - 4\Sigma_1^\infty \zeta'(\sin 2nx)\,q^{2n},$$
$$\rho\,\mathrm{zc}\,\rho x = -\tan x - 4\Sigma_1^\infty \zeta\ (\sin 2nx)\,q^{2n},$$
$$k'\rho\,\mathrm{sc}\,\rho x = \quad \tan x - 4\Sigma_1^\infty \xi\ (\sin 2nx)\,q^{2n}.$$

In each of the four groups the coefficient of the same power of q in the four series consists of the same sines or cosines, though the signs connecting them may be different. In each of the first three groups there are but two distinct coefficients, the other two differing from them, if at all, only by a change of sign of the whole coefficient. In the last group all the four coefficients are distinct in structure, except for uneven values of n when they all become identical.

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In each of the four groups two of the formulæ are convertible into the other two by the substitution of $x+\frac{1}{2}\pi$ for x; and in each of the first three groups two of the formulæ are convertible into the other two by a change of sign of q.

§ 10. If we dispense with the notation σ, ζ, ζ', &c. (§ 4), this system assumes the form:

$$k\rho\,\mathrm{sn}\,\rho x = 4\Sigma.\Sigma_m \sin\delta x.q^{\frac{1}{2}m},$$
$$kk'\rho\,\mathrm{sd}\,\rho x = 4\Sigma.\Sigma_m (-)^M \sin\delta x.q^{\frac{1}{2}m},$$
$$k\rho\,\mathrm{cd}\,\rho x = 4\Sigma.\Sigma_m (-)^\Delta \cos\delta x.q^{\frac{1}{2}m},$$
$$k\rho\,\mathrm{cn}\,\rho x = 4\Sigma.\Sigma_m (-)^{M+\Delta} \cos\delta x.q^{\frac{1}{2}m};$$

$$\rho\,\mathrm{zn}\,\rho x = \quad 4\Sigma.\Sigma_n \sin 2\delta' x.q^n,$$
$$\rho\,\mathrm{zd}\,\rho x = \quad 4\Sigma.\Sigma_n (-)^n \sin 2\delta' x.q^n,$$
$$\rho\,\mathrm{dn}\,\rho x = 1+4\Sigma.\Sigma_n (-)^\Delta \cos 2\delta' x.q^n,$$
$$k'\rho\,\mathrm{nd}\,\rho x = 1+4\Sigma.\Sigma_n (-)^{n+\Delta} \cos 2\delta' x.q^n;$$

$$\rho\,\mathrm{ns}\,\rho x = \mathrm{cec}\,x+4\Sigma.\Sigma_n \sin\delta x.q^n,$$
$$\rho\,\mathrm{ds}\,\rho x = \mathrm{cec}\,x+4\Sigma.\Sigma_n (-)^n \sin\delta x.q^n,$$
$$\rho\,\mathrm{dc}\,\rho x = \sec x+4\Sigma.\Sigma_n (-)^\Delta \cos\delta x.q^n,$$
$$k'\rho\,\mathrm{nc}\,\rho x = \sec x+4\Sigma.\Sigma_n (-)^{n+\Delta} \cos\delta x.q^n;$$

$$\rho\,\mathrm{zs}\,\rho x = \quad \cot x+4\Sigma.\Sigma_n \sin 2dx.q^{2n},$$
$$\rho\,\mathrm{cs}\,\rho x = \quad \cot x+4\Sigma.\Sigma_n (-)^{d'} \sin 2dx.q^{2n},$$
$$\rho\,\mathrm{zc}\,\rho x = -\tan x+4\Sigma.\Sigma_n (-)^d \sin 2dx.q^{2n},$$
$$k'\rho\,\mathrm{sc}\,\rho x = \quad \tan x-4\Sigma.\Sigma_n (-)^{d+d'} \sin 2dx.q^{2n}.$$

The exponents M and Δ denotes $\frac{1}{2}(m-1)$ and $\frac{1}{2}(\delta-1)$ respectively.

§ 11. In the preceding systems of formulæ some of the series involve sines and some involve cosines. In those involving cosines the sign-symbol $(-)^{\frac{1}{2}(\delta-1)}$ always occurs. We may therefore introduce greater regularity into the

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in this respect if we include the negative as well as the positive divisors of n in the following manner.

Let $d_1, d_2, \ldots, d_r$ be, as before, the positive divisors of n, and let $\Sigma'_n \phi(d)$ denote

$$\phi(d_1) + \ldots + \phi(d_r) - \phi(-d_1) - \phi(-d_2) - \ldots - \phi(-d_r),$$

$$\Sigma'_n \phi(d) = \Sigma_n \phi(d) - \Sigma_n \phi(-d).$$

Let similar meanings be assigned to $\Sigma'_n \phi(\delta)$ and $\Sigma'_n \phi(\delta')$. Then all the terms in $\Sigma'_n (-)^{\frac{1}{2}(\delta-1)} \phi(\delta)$ and in $\Sigma'_n (-)^{\frac{1}{2}(\delta-1)} \phi(\delta')$ have the same sign.

Denoting e^{ix} by z, we may write the system in the form

$$\begin{aligned}
k\rho \operatorname{sn} \rho x &= -2i\Sigma\Sigma'_m z^{\delta} q^{\frac{1}{2}m}, \\
kk'\rho \operatorname{sd} \rho x &= -2i\Sigma\Sigma'_m (-)^{\frac{1}{2}(m-1)} z^{\delta} q^{\frac{1}{2}m}, \\
k\rho \operatorname{cd} \rho x &= -2i\Sigma\Sigma'_m (-)^{\frac{1}{2}(\delta-1)} z^{\delta} q^{\frac{1}{2}m}, \\
k\rho \operatorname{cn} \rho x &= -2i\Sigma\Sigma'_m (-)^{\frac{1}{2}(m-1)+\frac{1}{2}(\delta-1)} z^{\delta} q^{\frac{1}{2}m};
\end{aligned}$$

$$\begin{aligned}
\rho \operatorname{zn} \rho x &= -2i\Sigma\Sigma'_n z^{2\delta'} q^{n}, \\
\rho \operatorname{zd} \rho x &= -2i\Sigma\Sigma'_n (-)^{n} z^{2\delta'} q^{n}, \\
\rho \operatorname{dn} \rho x &= 1 - 2i\Sigma\Sigma'_n (-)^{\frac{1}{2}(\delta-1)} z^{2\delta'} q^{n}, \\
k'\rho \operatorname{nd} \rho x &= 1 - 2i\Sigma\Sigma'_n (-)^{n+\frac{1}{2}(\delta-1)} z^{2\delta'} q^{n};
\end{aligned}$$

$$\begin{aligned}
\rho \operatorname{ns} \rho x &= \operatorname{cec} x - 2i\Sigma\Sigma_n z^{\delta} q^{n}, \\
\rho \operatorname{ds} \rho x &= \operatorname{cec} x - 2i\Sigma\Sigma_n (-)^{n} z^{\delta} q^{n}, \\
\rho \operatorname{dc} \rho x &= \sec x - 2i\Sigma\Sigma_n (-)^{\frac{1}{2}(\delta-1)} z^{\delta} q^{n}, \\
k'\rho \operatorname{nc} \rho x &= \sec x - 2i\Sigma\Sigma_n (-)^{n+\frac{1}{2}(\delta-1)} z^{\delta} q^{n};
\end{aligned}$$

$$\begin{aligned}
\rho \operatorname{zs} \rho x &= \cot x - 2i\Sigma\Sigma_n z^{2d} q^{2n}, \\
\rho \operatorname{cs} \rho x &= \cot x - 2i\Sigma\Sigma_n (-)^{d'} z^{2d} q^{2n}, \\
\rho \operatorname{zc} \rho x &= -\tan x - 2i\Sigma\Sigma_n (-)^{d} z^{2d} q^{2n}, \\
k'\rho \operatorname{sc} \rho x &= \tan x + 2i\Sigma\Sigma_n (-)^{d+d'} z^{2d} q^{2n}.
\end{aligned}$$

§ 12. Putting for z and q their values in terms of x and μ,

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the preceding equations assume the form:

$$k\rho\ \mathrm{sn}\,\rho x = 2i\Sigma\Sigma'_m e^{-(\frac{1}{2}m\mu+i\delta x)},$$
$$kk'\rho\ \mathrm{sd}\,\rho x = 2i\Sigma\Sigma'_m (-)^{\frac{1}{2}(m-1)} e^{-(\frac{1}{2}m\mu+i\delta x)},$$
$$k\rho\ \mathrm{cd}\,\rho x = 2i\Sigma\Sigma'_m (-)^{\frac{1}{2}(\delta-1)} e^{-(\frac{1}{2}m\mu+i\delta x)},$$
$$k\rho\ \mathrm{cn}\,\rho x = 2i\Sigma\Sigma'_m (-)^{\frac{1}{2}(m-1)+\frac{1}{2}(\delta-1)} e^{-(\frac{1}{2}m\mu+i\delta x)};$$
$$\&c. \qquad\qquad \&c.$$

§ 13. By substituting ix for x in the formulæ of § 7 and changing the modulus from k to k' we may deduce the following system, in which ρ' and q' denote the same functions of k' that ρ and q are of k:

$$k'\rho'\ \mathrm{sc}\,\rho' x = 4\Sigma.\Sigma_m \sinh\delta x.q'^{\frac{1}{2}m},$$
$$kk'\rho'\ \mathrm{sd}\,\rho' x = 4\Sigma.\Sigma_m (-)^M \sinh\delta x.q'^{\frac{1}{2}m},$$
$$k'\rho'\ \mathrm{nd}\,\rho' x = 4\Sigma.\Sigma_m (-)^\Delta \cosh\delta x.q'^{\frac{1}{2}m},$$
$$k'\rho'\ \mathrm{nc}\,\rho' x = 4\Sigma.\Sigma_m (-)^{M+\Delta} \cosh\delta x.q'^{\frac{1}{2}m};$$

$$\rho'\ \mathrm{zc}\,\rho' x = -\frac{2x}{\mu'} - 4\Sigma.\Sigma_n \sinh 2\delta' x.q'^n,$$
$$\rho'\ \mathrm{zd}\,\rho' x = -\frac{2x}{\mu'} - 4\Sigma.\Sigma_n (-)^n \sinh 2\delta' x.q'^n,$$
$$\rho'\ \mathrm{dc}\,\rho' x = 1 + 4\Sigma.\Sigma_n (-)^\Delta \cosh 2\delta' x.q'^n,$$
$$k\rho'\ \mathrm{cd}\,\rho' x = 1 + 4\Sigma.\Sigma_n (-)^{n+\Delta} \cosh 2\delta' x.q'^n;$$

$$\rho'\ \mathrm{cs}\,\rho' x = \mathrm{cech}\,x - 4\Sigma.\Sigma_n \sinh\delta x.q'^n,$$
$$\rho'\ \mathrm{ds}\,\rho' x = \mathrm{cech}\,x - 4\Sigma.\Sigma_n (-)^n \sinh\delta x.q'^n,$$
$$\rho'\ \mathrm{dn}\,\rho' x = \mathrm{sech}\,x + 4\Sigma.\Sigma_n (-)^\Delta \cosh\delta x.q'^n,$$
$$k\rho'\ \mathrm{cn}\,\rho' x = \mathrm{sech}\,x + 4\Sigma.\Sigma_n (-)^{n+\Delta} \cosh\delta x.q'^n;$$

$$\rho'\ \mathrm{zs}\,\rho' x = -\frac{2x}{\mu'} + \coth x - 4\Sigma.\Sigma_n \sinh 2dx.q'^{2n},$$
$$\rho'\ \mathrm{ns}\,\rho' x = \coth x - 4\Sigma.\Sigma_n (-)^{d'} \sin 2dx.q'^{2n},$$
$$\rho'\ \mathrm{zn}\,\rho' x = -\frac{2x}{\mu'} + \tanh x - 4\Sigma.\Sigma_n (-)^d \sin 2dx.q'^{2n},$$
$$k\rho'\ \mathrm{sn}\,\rho' x = \tanh x + 4\Sigma.\Sigma_n (-)^{d+d'} \sin 2dx.q'^{2n}.$$

In the above system the formulæ have been arranged in groups depending upon the forms of the series.

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§ 14. We may express the preceding system in forms analogous to those given in §§ 10 and 11.

Taking for example the first group, we have

$$k'\rho' \operatorname{sc}\rho' x = 2\Sigma . \Sigma'_m e^{\delta x} . q'^{\frac{1}{4}m},$$
$$kk'\rho' \operatorname{sd}\rho' x = 2\Sigma . \Sigma'_m (-)^{\frac{1}{2}(m-1)} e^{\delta x} . q'^{\frac{1}{4}m},$$
$$k'\rho' \operatorname{nd}\rho' x = 2\Sigma . \Sigma'_m (-)^{\frac{1}{2}(\delta-1)} e^{\delta x} . q'^{\frac{1}{4}m},$$
$$k'\rho' \operatorname{nc}\rho' x = 2\Sigma . \Sigma'_m (-)^{\frac{1}{2}(m-1)+\frac{1}{2}(\delta-1)} e^{\delta x} . q'^{\frac{1}{4}m};$$
$$\&c. \qquad\qquad \&c.$$

and

$$k'\rho' \operatorname{sc}\rho' x = 2\Sigma . \Sigma'_m e^{-\frac{1}{2}m\mu' + \delta x},$$
$$kk'\rho' \operatorname{sd}\rho' x = 2\Sigma . \Sigma'_m (-)^{\frac{1}{2}(m-1)} e^{-\frac{1}{2}m\mu' + \delta x},$$
$$k'\rho' \operatorname{nd}\rho' x = 2\Sigma . \Sigma'_m (-)^{\frac{1}{2}(\delta-1)} e^{-\frac{1}{2}m\mu' + \delta x},$$
$$k'\rho' \operatorname{nc}\rho' x = 2\Sigma . \Sigma'_m (-)^{\frac{1}{2}(m-1)+\frac{1}{2}(\delta-1)} e^{-\frac{1}{2}m\mu' + \delta x};$$
$$\&c. \qquad\qquad \&c.$$

Series of the form $\Sigma_{-\infty}^{\infty} \phi(x + na)$, §§ 15, 16.

§ 15. In Trigonometry the four functions, the cotangent, cosecant, tangent, and secant may be expressed as infinite series of simple reciprocals of the forms $(x + n\pi)^{-1}$ and $(x + \frac{1}{2}m\pi)^{-1}$, viz. we have

$$\cot x = \Sigma_{-\infty}^{\infty} \frac{1}{x + n\pi}, \qquad \operatorname{cec} x = \Sigma_{-\infty}^{\infty} (-)^n \frac{1}{x + n\pi},$$

$$\tan x = -\Sigma_{-\infty}^{\infty} \frac{1}{x + \frac{1}{2}m\pi}, \qquad \sec x = \Sigma_{-\infty}^{\infty} (-)^M \frac{1}{x + \frac{1}{2}m\pi}.$$

The sixteen elliptic and Zeta functions $\rho \operatorname{ns}\rho x$, &c. are derivable from the four circular functions $\cot x$, $\operatorname{cec} x$, $\tan x$, $\sec x$ by a method of treatment exactly similar to that by which the latter four functions are derivable from a simple reciprocal x^{-1}.

The formulæ in question are as follows:

$$\rho \operatorname{ns}\rho x = \Sigma_{-\infty}^{\infty} \qquad \operatorname{cec}(x + ni\mu),$$
$$\rho \operatorname{ds}\rho x = \Sigma_{-\infty}^{\infty} (-)^n \operatorname{cec}(x + ni\pi),$$
$$\rho \operatorname{cs}\rho x = \Sigma_{-\infty}^{\infty} (-)^n \cot(x + ni\mu),$$
$$\rho \operatorname{zs}\rho x = \Sigma_{-\infty}^{\infty} \qquad \cot(x + ni\mu);$$

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$$\begin{aligned}
\rho\,\mathrm{dc}\,\rho x &= \Sigma_{-\infty}^{\infty} \sec(x+ni\mu),\\
k'\rho\,\mathrm{nc}\,\rho x &= \Sigma_{-\infty}^{\infty} (-)^n \sec(x+ni\mu),\\
k'\rho\,\mathrm{sc}\,\rho x &= \Sigma_{-\infty}^{\infty} (-)^n \tan(x+ni\mu),\\
\rho\,\mathrm{zc}\,\rho x &= -\Sigma_{-\infty}^{\infty} \tan(x+ni\mu);
\end{aligned}$$

$$\begin{aligned}
k\rho\,\mathrm{cd}\,\rho x &= \Sigma_{-\infty}^{\infty} \sec(x+\tfrac{1}{2}mi\mu),\\
kk'\rho\,\mathrm{sd}\,\rho x &= -i\Sigma_{-\infty}^{\infty} (-)^M \sec(x+\tfrac{1}{2}mi\mu),\\
k'\rho\,\mathrm{nd}\,\rho x &= 1-i\Sigma_{-\infty}^{\infty} (-)^M \tan(x+\tfrac{1}{2}mi\mu),\\
\rho\,\mathrm{zd}\,\rho x &= -\Sigma_{-\infty}^{\infty} \tan(x+\tfrac{1}{2}mi\mu);
\end{aligned}$$

$$\begin{aligned}
k\rho\,\mathrm{sn}\,\rho x &= \Sigma_{-\infty}^{\infty} \operatorname{cec}(x+\tfrac{1}{2}mi\mu),\\
k\rho\,\mathrm{cn}\,\rho x &= i\Sigma_{-\infty}^{\infty} (-)^M \operatorname{cec}(x+\tfrac{1}{2}mi\mu),\\
\rho\,\mathrm{dn}\,\rho x &= 1+i\Sigma_{-\infty}^{\infty} (-)^M \cot(x+\tfrac{1}{2}mi\mu),\\
\rho\,\mathrm{zn}\,\rho x &= \Sigma_{-\infty}^{\infty} \cot(x+\tfrac{1}{2}mi\mu).
\end{aligned}$$

In the preceding system of formulæ n is to have the values $0, \pm 1, \pm 2, \ldots \pm v$, v being an infinite integer: and m is to have the values $\pm 1, \pm 3, \ldots \pm w$, w being an infinite uneven integer. In the series for $k'\rho\,\mathrm{nd}\,\rho x$ and $\rho\,\mathrm{dn}\,\rho x$ the limiting value w is supposed to be of the form $4n+3$: if it be of the form $4n+1$, the term 1 is to be replaced by -1.

§ 16. Transforming the above formulæ by the substitution of ix for x, we obtain a corresponding system in which the sixteen functions are expressed as series of hyperbolic cotangents, cosecants, &c. In these formulæ n and m have the same range of values as in the last section.

The restriction upon the value of w in the case of the series for $k'\rho\,\mathrm{nd}\,\rho x$ and $\rho\,\mathrm{dn}\,\rho x$ is no longer necessary: but a similar restriction is required in the case of the series $\rho'\,\mathrm{dc}\,\rho' x$ and $k\rho'\,\mathrm{cd}\,\rho' x$, viz. the first term in these series is 1 or -1, according as w is of the form $4n+3$ or $4n+1$.

This system of formulæ is as follows:

$$\begin{aligned}
\rho'\,\mathrm{ns}\,\rho' x &= \Sigma_{-\infty}^{\infty} (-)^n \coth(x+n\mu'),\\
\rho'\,\mathrm{ds}\,\rho' x &= \Sigma_{-\infty}^{\infty} (-)^n \operatorname{cech}(x+n\mu'),\\
\rho'\,\mathrm{cs}\,\rho' x &= \Sigma_{-\infty}^{\infty} \operatorname{cech}(x+n\mu'),\\
\rho'\,\mathrm{zs}\,\rho' x &= -\frac{2x}{\mu'} + \Sigma_{-\infty}^{\infty} \coth(x+n\mu');
\end{aligned}$$

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$$\rho' \operatorname{dc}\rho' x = 1 + \Sigma_{-\infty}^{\infty} (-)^M \coth(x + \tfrac{1}{2}m\mu'),$$
$$k'\rho' \operatorname{nc}\rho' x = \Sigma_{-\infty}^{\infty} (-)^M \operatorname{cech}(x + \tfrac{1}{2}m\mu'),$$
$$k'\rho' \operatorname{sc}\rho' x = -\Sigma_{-\infty}^{\infty} \operatorname{cech}(x + \tfrac{1}{2}m\mu'),$$
$$\rho' \operatorname{zc}\rho' x = -\frac{2x}{\mu'} + \Sigma_{-\infty}^{\infty} \coth(x + \tfrac{1}{2}m\mu');$$

$$k\rho' \operatorname{cd}\rho' x = 1 + \Sigma_{-\infty}^{\infty} (-)^M \tanh(x + \tfrac{1}{2}m\mu'),$$
$$kk'\rho' \operatorname{sd}\rho' x = -\Sigma_{-\infty}^{\infty} (-)^M \operatorname{sech}(x + \tfrac{1}{2}m\mu'),$$
$$k'\rho' \operatorname{nd}\rho' x = \Sigma_{-\infty}^{\infty} \operatorname{sech}(x + \tfrac{1}{2}m\mu'),$$
$$\rho' \operatorname{zd}\rho' x = -\frac{2x}{\mu'} + \Sigma_{-\infty}^{\infty} \tanh(x + \tfrac{1}{2}m\mu');$$

$$k\rho' \operatorname{sn}\rho' x = \Sigma_{-\infty}^{\infty} (-)^n \tanh(x + n\mu'),$$
$$k\rho' \operatorname{cn}\rho' x = \Sigma_{-\infty}^{\infty} (-)^n \operatorname{sech}(x + n\mu'),$$
$$\rho' \operatorname{dn}\rho' x = \Sigma_{-\infty}^{\infty} \operatorname{sech}(x + n\mu'),$$
$$\rho' \operatorname{zn}\rho' x = -\frac{2x}{\mu'} + \Sigma_{-\infty}^{\infty} \tanh(x + n\mu').$$

The q-series, §§ 17–25.

§ 17. In the original form of the q-series for the sixteen elliptic and Zeta functions given in vol. XVII., pp. 2, 3 the coefficients are all of the form $\frac{q^{\frac{1}{2}r}}{1 \pm q^r}$ and $\frac{q^r}{1 \pm q^r}$. It is obvious that the former expression represents the hyperbolic secant or cosecant of $\frac{1}{2}r\mu$; but is is not quite so evident that the latter expression is the representative of the hyperbolic tangent and cotangent: in fact we have

$$\frac{2q^{\frac{1}{2}r}}{1 + q^r} = \operatorname{sech} \tfrac{1}{2}r\mu,$$
$$\frac{2q^{\frac{1}{2}r}}{1 - q^r} = \operatorname{cech} \tfrac{1}{2}r\mu,$$
$$\frac{2q^r}{1 + q^r} = 1 - \tanh \tfrac{1}{2}r\mu,$$
$$\frac{2q^r}{1 - q^r} = -1 + \coth \tfrac{1}{2}r\mu.$$

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§ 18. If in any q-series in which the coefficients are of the two latter forms we separate the term unity from $\tanh \frac{1}{2}r\mu$ or $\coth \frac{1}{2}r\mu$ in each coefficient, so as to form two series in one of which the coefficients are unity, while in the other they depend upon μ, it is evident that these two series are both neutral, but that their difference, obtained by combining corresponding terms, is convergent. Using the letters v and w as in the last two sections to denote an infinitely large number, and an infinitely large uneven number, and summing the series in which the coefficients are unity, we obtain the following system:

$$\rho \operatorname{ns}\rho x = \frac{\cos(w+1)x}{\sin x} + 2\,\Sigma_1^w \coth \tfrac{1}{2}m\mu \sin mx,$$

$$\rho \operatorname{ds}\rho x = \frac{\cos(w+1)x}{\sin x} + 2\,\Sigma_1^w \tanh \tfrac{1}{2}m\mu \sin mx,$$

$$\rho \operatorname{cs}\rho x = \frac{\cos(2v+1)x}{\sin x} + 2\,\Sigma_1^v \tanh n\mu \sin 2nx,$$

$$\rho \operatorname{zs}\rho x = \frac{\cos(2v+1)x}{\sin x} + 2\,\Sigma_1^v \coth n\mu \sin 2nx;$$

$$\rho \operatorname{dc}\rho x = (-)^{\frac{1}{2}(w+1)} \frac{\cos(w+1)x}{\cos x} + 2\,\Sigma_1^w (-)^M \coth \tfrac{1}{2}m\mu \cos mx,$$

$$k'\rho \operatorname{nc}\rho x = (-)^{\frac{1}{2}(w+1)} \frac{\cos(w+1)x}{\cos x} + 2\,\Sigma_1^w (-)^M \tanh \tfrac{1}{2}m\mu \cos mx,$$

$$k'\rho \operatorname{sc}\rho x = (-)^v \frac{\sin(2v+1)x}{\cos x} - 2\,\Sigma_1^v (-)^n \tanh n\mu \sin 2nx,$$

$$\rho \operatorname{zc}\rho x = (-)^{v+1} \frac{\sin(2v+1)x}{\cos x} + 2\,\Sigma_1^v (-)^n \coth n\mu \sin 2nx;$$

$$k\rho \operatorname{cd}\rho x = 2\,\Sigma_1^\infty (-)^M \operatorname{cech} \tfrac{1}{2}m\mu \cos mx,$$

$$kk'\rho \operatorname{sd}\rho x = 2\,\Sigma_1^\infty (-)^M \operatorname{sech} \tfrac{1}{2}m\mu \sin mx,$$

$$k'\rho \operatorname{nd}\rho x = 1 + 2\,\Sigma_1^\infty (-)^n \operatorname{sech} n\mu \cos 2nx,$$

$$\rho \operatorname{zd}\rho x = 2\,\Sigma_2^\infty (-)^n \operatorname{cech} n\mu \sin 2nx;$$

$$k\rho \operatorname{sn}\rho x = 2\,\Sigma_1^\infty \operatorname{cech} \tfrac{1}{2}m\mu \sin mx,$$

$$k\rho \operatorname{cn}\rho x = 2\,\Sigma_1^\infty \operatorname{sech} \tfrac{1}{2}m\mu \cos mx,$$

$$\rho \operatorname{dn}\rho x = 1 + 2\,\Sigma_1^\infty \operatorname{sech} n\mu \cos 2nx,$$

$$\rho \operatorname{zn}\rho x = 2\,\Sigma_1^\infty \operatorname{cech} n\mu \sin 2nx.$$

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§ 19. The last two groups may be expressed in a slightly different form by taking the limits of the summation to be $\pm\infty$. The term unity which occurs in the expressions for $k'\rho\,\mathrm{nd}\,\rho x$ and $\rho\,\mathrm{dn}\,\rho x$ then disappears, as it corresponds to the value $n=0$. But if the value $n=0$ be included in the Zeta series it is necessary to add the term $-\frac{2x}{\mu}$ in order to cancel the positive term of this magnitude which then occurs under the sign of summation. Thus taking, for example, the last group, and, including the value $n=0$, we have

$$k\rho\,\mathrm{sn}\,\rho x = \Sigma_{-\infty}^{\infty}\frac{\sin mx}{\sinh\frac{1}{2}m\mu},$$

$$k\rho\,\mathrm{cn}\,\rho x = \Sigma_{-\infty}^{\infty}\frac{\cos mx}{\cosh\frac{1}{2}m\mu},$$

$$\rho\,\mathrm{dn}\,\rho x = \Sigma_{-\infty}^{\infty}\frac{\cos 2nx}{\cosh n\mu},$$

$$\rho\,\mathrm{zn}\,\rho x = -\frac{2x}{\mu} + \Sigma_{-\infty}^{\infty}\frac{\sin 2nx}{\sinh n\mu}.$$

§ 20. The complete system of formulæ may however be more properly expressed by a summation having reference to the powers of e^{ix}. Denoting e^{ix} by z, as in § 11, we have

$$\rho\,\mathrm{ns}\,\rho x = \frac{\cos(w+1)x}{\sin x} - i\,\Sigma_{-w}^{w}\coth\tfrac{1}{2}m\mu.z^m,$$

$$\rho\,\mathrm{ds}\,\rho x = \frac{\cos(w+1)x}{\sin x} - i\,\Sigma_{-w}^{w}\tanh\tfrac{1}{2}m\mu.z^m,$$

$$\rho\,\mathrm{cs}\,\rho x = \frac{\cos(2v+1)x}{\sin x} - i\,\Sigma_{-v}^{v}\tanh n\mu.z^{2n},$$

$$\rho\,\mathrm{zs}\,\rho x = \frac{\cos(2v+1)x}{\sin x} - i\,\Sigma_{-v}^{v}\coth n\mu.z^{2n};$$

$$\rho\,\mathrm{dc}\,\rho x = (-)^{\frac{1}{2}(w+1)}\frac{\cos(w+1)x}{\cos x} + \Sigma_{-w}^{w}(-)^M\coth\tfrac{1}{2}m\mu.z^m,$$

$$k'\rho\,\mathrm{nc}\,\rho x = (-)^{\frac{1}{2}(w+1)}\frac{\cos(w+1)x}{\cos x} + \Sigma_{-w}^{w}(-)^M\tanh\tfrac{1}{2}m\mu.z^m,$$

$$k'\rho\,\mathrm{sc}\,\rho x = (-)^{v}\frac{\sin(2v+1)x}{\cos x} + i\,\Sigma_{-v}^{v}(-)^n\tanh n\mu.z^{2n},$$

$$\rho\,\mathrm{zc}\,\rho x = (-)^{v+1}\frac{\sin(2v+1)x}{\cos x} - i\,\Sigma_{-v}^{v}(-)^n\coth n\mu.z^{2n};$$

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$$k\rho \operatorname{cd}\rho x = \Sigma_{-\infty}^{\infty}(-)^{M} \operatorname{cech}\tfrac{1}{2}m\mu . z^{m},$$
$$kk'\rho \operatorname{sd}\rho x = -i\Sigma_{-\infty}^{\infty}(-)^{M} \operatorname{sech}\tfrac{1}{2}m\mu . z^{m},$$
$$k'\rho \operatorname{nd}\rho x = \Sigma_{-\infty}^{\infty}(-)^{n} \operatorname{sech} n\mu . z^{2m},$$
$$\rho \operatorname{zd}\rho x = -i\Sigma_{-\infty}^{\infty}(-)^{n} \operatorname{cech} n\mu . z^{2m};$$

$$k\rho \operatorname{sn}\rho x = -i\Sigma_{-\infty}^{\infty} \operatorname{cech}\tfrac{1}{2}m\mu . z^{m},$$
$$k\rho \operatorname{cn}\rho x = \Sigma_{-\infty}^{\infty} \operatorname{sech}\tfrac{1}{2}m\mu . z^{m},$$
$$\rho \operatorname{dn}\rho x = \Sigma_{-\infty}^{\infty} \operatorname{sech} n\mu . z^{2m},$$
$$\rho \operatorname{zn}\rho x = -i\Sigma_{-\infty}^{\infty} \operatorname{cech} n\mu . z^{2m}.$$

In the case of the four Zeta functions the value $n=0$ is supposed to be excluded.

Thus the sixteen elliptic and Zeta functions may be regarded as the representatives of the series

$$\Sigma_{-\infty}^{\infty} A_{2m} e^{2mix}, \quad \Sigma_{-\infty}^{\infty} \pm A_{2m} e^{2mix}, \quad \Sigma_{-\infty}^{\infty} A_{m} e^{mix}, \quad \Sigma_{-\infty}^{\infty} \pm A_{m} e^{mix},$$

where, if a be a given constant, the coefficient A_p denotes the hyperbolic tangent, cotangent, secant, or cosecant of pa.

§ 21. It may be remarked that the last group of formulæ may be expressed in the following form, the value $n=0$ being excluded in the fourth series,

$$k\rho \operatorname{sn}\rho x = i\Sigma_{-\infty}^{\infty} \frac{z^{m}}{\sin\tfrac{1}{2}mi\mu},$$
$$k\rho \operatorname{cn}\rho x = \Sigma_{-\infty}^{\infty} \frac{z^{m}}{\cos\tfrac{1}{2}mi\mu},$$
$$\rho \operatorname{dn}\rho x = \Sigma_{-\infty}^{\infty} \frac{z^{2m}}{\cos ni\mu},$$
$$\rho \operatorname{zn}\rho x = \Sigma_{-\infty}^{\infty} \frac{z^{2m}}{\sin ni\mu}.$$

The previous group may of course be similarly expressed, the only difference in the series being the occurrence of the factors $(-)^{M}$ and $(-)^{n}$ in the first two and the last two respectively.

§ 22. By transforming the q-series by putting ix for x and changing the modulus from k to k', we obtain the following system of series proceeding by hyperbolic sines and cosines of multiples of x:

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$$\rho' \operatorname{ns}\rho' x = \coth x + \Sigma_1^\infty \frac{4q'^{2n}}{1+q'^{2n}} \sinh 2nx,$$

$$\rho' \operatorname{ds}\rho' x = \operatorname{cech} x + \Sigma_1^\infty \frac{4q'^{2n-1}}{1+q'^{2n-1}} \sinh(2n-1)x,$$

$$\rho' \operatorname{cs}\rho' x = \operatorname{cech} x - \Sigma_1^\infty \frac{4q'^{2n-1}}{1-q'^{2n-1}} \sinh(2n-1)x,$$

$$\rho' \operatorname{zs}\rho' x = -\frac{2x}{\mu'} + \coth x - \Sigma_1^\infty \frac{4q'^{2n}}{1-q'^{2n}} \sinh 2nx;$$

$$\rho' \operatorname{dc}\rho' x = 1 + \Sigma_1^\infty \frac{4q'^{n}}{1+q'^{2n}} \cosh 2nx,$$

$$k'\rho' \operatorname{nc}\rho' x = \Sigma_1^\infty \frac{4q'^{\frac{1}{2}(2n-1)}}{1+q'^{2n-1}} \cosh(2n-1)x,$$

$$k'\rho' \operatorname{sc}\rho' x = \Sigma_1^\infty \frac{4q'^{\frac{1}{2}(2n-1)}}{1-q'^{2n-1}} \sinh(2n-1)x,$$

$$\rho' \operatorname{zc}\rho' x = -\frac{2x}{\mu'} - \Sigma_1^\infty \frac{4q'^{n}}{1-q'^{2n}} \sinh 2nx;$$

$$k\rho' \operatorname{cd}\rho' x = 1 + \Sigma_1^\infty (-)^n \frac{4q'^{n}}{1+q'^{2n}} \cosh 2nx,$$

$$kk'\rho' \operatorname{sd}\rho' x = -\Sigma_1^\infty (-)^n \frac{4q'^{\frac{1}{2}(2n-1)}}{1+q'^{2n-1}} \sinh(2n-1)x,$$

$$k'\rho' \operatorname{nd}\rho' x = -\Sigma_1^\infty (-)^n \frac{4q'^{\frac{1}{2}(2n-1)}}{1-q'^{2n-1}} \cosh(2n-1)x,$$

$$\rho' \operatorname{zd}\rho' x = -\frac{2x}{\mu'} - \Sigma_1^\infty (-)^n \frac{4q'^{n}}{1-q'^{2n}} \sinh 2nx;$$

$$k\rho' \operatorname{sn}\rho' x = \tanh x + \Sigma_1^\infty (-)^n \frac{4q'^{n}}{1+q'^{2n}} \sinh 2nx,$$

$$k\rho' \operatorname{cn}\rho' x = \operatorname{sech} x + \Sigma_1^\infty (-)^n \frac{4q'^{2n-1}}{1+q'^{2n-1}} \cosh(2n-1)x,$$

$$\rho' \operatorname{dn}\rho' x = \operatorname{sech} x - \Sigma_1^\infty (-)^n \frac{4q'^{2n-1}}{1-q'^{2n-1}} \cosh(2n-1)x,$$

$$\rho' \operatorname{zn}\rho' x = -\frac{2x}{\mu'} + \tanh x - \Sigma_1^\infty (-)^n \frac{4q'^{2n}}{1-q'^{2n}} \sinh 2nx.$$

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§ 23. Expressing the q'-coefficients as hyperbolic functions of μ' (§17), and replacing the hyperbolic functions of multiples of x by their exponential values, we obtain the following formulæ which correspond to those in § 20. The value $n=0$ is supposed to be excluded from the four Zeta series.

$$\rho'\,\mathrm{ns}\,\rho' x = \tfrac{1}{2}e^{(2v+1)x}\operatorname{cech} x - \Sigma_{-v}^{v}\tanh n\mu'.e^{2nx},$$

$$\rho'\,\mathrm{ds}\,\rho' x = \tfrac{1}{2}e^{(w+1)x}\operatorname{cech} x - \Sigma_{-w}^{w}\tanh \tfrac{1}{2}m\mu'.e^{mx},$$

$$\rho'\,\mathrm{cs}\,\rho' x = \tfrac{1}{2}e^{(w+1)x}\operatorname{cech} x - \Sigma_{-w}^{w}\coth \tfrac{1}{2}m\mu'.e^{mx},$$

$$\rho'\,\mathrm{zs}\,\rho' x = -\frac{2x}{\mu'} + \tfrac{1}{2}e^{(2v+1)}\operatorname{cech} x - \Sigma_{-v}^{v}\coth n\mu'.e^{2nx};$$

$$\rho'\,\mathrm{dc}\,\rho' x = \Sigma_{-\infty}^{\infty}\operatorname{sech} n\mu'.e^{2nx},$$

$$k'\rho'\,\mathrm{nc}\,\rho' x = \Sigma_{-\infty}^{\infty}\operatorname{sech} \tfrac{1}{2}m\mu'.e^{mx},$$

$$k'\rho'\,\mathrm{sc}\,\rho' x = \Sigma_{-\infty}^{\infty}\operatorname{cech} \tfrac{1}{2}m\mu'.e^{mx},$$

$$\rho'\,\mathrm{zc}\,\rho' x = -\frac{2x}{\mu'} - \Sigma_{-\infty}^{\infty}\operatorname{cech} n\mu'.e^{2nx};$$

$$k\rho'\,\mathrm{cd}\,\rho' x = \Sigma_{-\infty}^{\infty}(-)^{n}\operatorname{sech} n\mu'.e^{2nx},$$

$$kk'\rho'\,\mathrm{sd}\,\rho' x = \Sigma_{-\infty}^{\infty}(-)^{M}\operatorname{sech} \tfrac{1}{2}m\mu'.e^{mx},$$

$$k'\rho'\,\mathrm{nd}\,\rho' x = \Sigma_{-\infty}^{\infty}(-)^{M}\operatorname{cech} \tfrac{1}{2}m\mu'.e^{mx},$$

$$\rho'\,\mathrm{zd}\,\rho' x = -\frac{2x}{\mu'} - \Sigma_{-\infty}^{\infty}(-)^{n}\operatorname{cech} n\mu'.e^{2nx};$$

$$k\rho'\,\mathrm{sn}\,\rho' x = \tfrac{1}{2}(-)^{v}e^{(2v+1)x}\operatorname{sech} x - \Sigma_{-v}^{v}(-)^{n}\tanh n\mu'.e^{2nx},$$

$$k\rho'\,\mathrm{cn}\,\rho' x = \tfrac{1}{2}(-)^{\frac{1}{2}(w+1)}e^{(w+1)x}\operatorname{sech} x + \Sigma_{-w}^{w}(-)^{M}\tanh \tfrac{1}{2}m\mu'.e^{mx},$$

$$\rho'\,\mathrm{dn}\,\rho' x = \tfrac{1}{2}(-)^{\frac{1}{2}(w+1)}e^{(w+1)x}\operatorname{sech} x + \Sigma_{-w}^{w}(-)^{M}\coth \tfrac{1}{2}m\mu'.e^{mx},$$

$$\rho'\,\mathrm{zn}\,\rho' x = -\frac{2x}{\mu'} + \tfrac{1}{2}(-)^{v}e^{(2v+1)}\operatorname{sech} x - \Sigma_{-v}^{v}(-)^{n}\coth n\mu'.e^{2nx}.$$

In the first and last groups the functions $\cosh(2v+1)x$, $\cosh(w+1)x$, &c. have been replaced by the exponentials $\frac{1}{2}e^{(2v+1)x}$, $\frac{1}{2}e^{(w+1)x}$, &c., to which they are equivalent since v and w are infinite.

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§ 24. The following system of expressions for $\rho' \operatorname{ns}\rho' x$, &c. seems also deserving of notice. The value $n = 0$ is supposed to be included in all the series in which n occurs.

$$\rho' \operatorname{ns}\rho' x = \Sigma_{-\infty}^{\infty} \frac{\cosh 2n\,(\tfrac{1}{2}\mu' - x)}{\cosh n\mu'},$$

$$\rho' \operatorname{ds}\rho' x = \Sigma_{-\infty}^{\infty} \frac{\cosh m\,(\tfrac{1}{2}\mu' - x)}{\cosh \tfrac{1}{2}m\mu'},$$

$$\rho' \operatorname{cs}\rho' x = \Sigma_{-\infty}^{\infty} \frac{\sinh m\,(\tfrac{1}{2}\mu' - x)}{\sinh \tfrac{1}{2}m\mu'},$$

$$\rho' \operatorname{zs}\rho' x = \Sigma_{-\infty}^{\infty} \frac{\sinh 2n\,(\tfrac{1}{2}\mu' - x)}{\sinh n\mu'};$$

$$\rho' \operatorname{dc}\rho' x = \Sigma_{-\infty}^{\infty} \frac{\cosh 2nx}{\cosh n\mu'},$$

$$k'\rho' \operatorname{nc}\rho' x = \Sigma_{-\infty}^{\infty} \frac{\cosh mx}{\cosh \tfrac{1}{2}m\mu'},$$

$$k'\rho' \operatorname{sc}\rho' x = \Sigma_{-\infty}^{\infty} \frac{\sinh mx}{\sinh \tfrac{1}{2}m\mu'},$$

$$\rho' \operatorname{zc}\rho' x = -\Sigma_{-\infty}^{\infty} \frac{\sinh 2nx}{\sinh n\mu'};$$

$$k\rho' \operatorname{cd}\rho' x = \Sigma_{-\infty}^{\infty} (-)^n \frac{\cosh 2nx}{\cosh n\mu'},$$

$$kk'\rho' \operatorname{sd}\rho' x = \Sigma_{-\infty}^{\infty} (-)^M \frac{\sinh mx}{\cosh \tfrac{1}{2}m\mu'},$$

$$k'\rho' \operatorname{nd}\rho' x = \Sigma_{-\infty}^{\infty} (-)^M \frac{\cosh mx}{\sinh \tfrac{1}{2}m\mu'},$$

$$\rho' \operatorname{zd}\rho' x = -\Sigma_{-\infty}^{\infty} (-)^n \frac{\sinh 2nx}{\sinh n\mu'};$$

$$k\rho' \operatorname{sn}\rho' x = \Sigma_{-\infty}^{\infty} (-)^n \frac{\cosh 2n\,(\tfrac{1}{2}\mu' - x)}{\cosh n\mu'},$$

$$k\rho' \operatorname{cn}\rho' x = \Sigma_{-\infty}^{\infty} (-)^M \frac{\sinh m\,(\tfrac{1}{2}\mu' - x)}{\cosh \tfrac{1}{2}m\mu'},$$

$$\rho' \operatorname{dn}\rho' x = \Sigma_{-\infty}^{\infty} (-)^M \frac{\cosh m\,(\tfrac{1}{2}\mu' - x)}{\sinh \tfrac{1}{2}m\mu'},$$

$$\rho' \operatorname{zn}\rho' x = -\Sigma_{-\infty}^{\infty} (-)^n \frac{\sinh 2n\,(\tfrac{1}{2}\mu' - x)}{\sinh n\mu'}.$$

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In the preceding system of formulæ we may evidently replace the hyperbolic sines and cosines in the numerators by simple exponentials, viz.

$$\rho' \operatorname{ns}\rho' x = \Sigma_{-\infty}^{\infty} \frac{e^{2n(\frac{1}{2}\mu' - x)}}{\cosh n\mu'}, \quad \&c.,$$

$$\rho' \operatorname{dc}\rho' x = \Sigma_{-\infty}^{\infty} \frac{e^{2nx}}{\cosh n\mu'}, \quad \&c.,$$

$$\&c., \qquad \&c.$$

§ 25. The series in §§ 15 and 20 (*i.e.* the series equivalent to the q-series) are convergent for all real values of x and μ. The series in §§ 22–24 are only convergent within restricted values of x: viz. in the first and last groups of each system x must lie between 0 and μ' and, in the two middle groups, between $\pm \frac{1}{2}\mu'$.

The conditions for the convergence of the series in the general case, when x and μ are complex, may be easily obtained by considering the series in the original q-form and q'-form (§ 22), and determining the limiting value of the general term when n or m is infinite.

Series for the elliptic and Zeta functions in powers of x, the coefficients being expressed in terms of h, §§ 26–28.

§ 26. The first four terms of the expansions of the elliptic and Zeta functions have been given in vol. XVII. pp. 10–12.

The coefficients of the general terms in the expansions of the twelve elliptic functions depend upon three functions which I have denoted by $\phi_n(a, b)$, $\psi_n(a, b)$, and $\chi_n(a, b)$, and the corresponding coefficients in the case of the Zeta functions depend upon two other functions which I have denoted by $f_n(a, b)$ and $F_n(a, b)$. The five quantities $\phi_n(a, b)$, &c. are homogeneous functions of a and b of the order n.

§ 27. The expansions of the sixteen functions when expressed by means of these five coefficients assume the following forms, in which, for simplicity of expression, x_n is used to denote $\frac{x^n}{n\,\mathrm{l}}$.

$$\operatorname{ns} x = x^{-1} + \Sigma_1^{\infty} \chi_n(1, h)\, x_{2n-1},$$
$$\operatorname{ds} x = x^{-1} + \Sigma_1^{\infty} \chi_n(h', -h)\, x_{2n-1},$$
$$\operatorname{cs} x = x^{-1} + \Sigma_1^{\infty} (-)^n \chi_n(1, h')\, x_{2n-1},$$
$$\operatorname{zs} x = x^{-1} - \tfrac{1}{3}(\mathrm{i} + \mathrm{g} + \mathrm{e})\, x - \Sigma_1^{\infty} F_{n-1}(1, h)\, x_{2n+1};$$

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$$\begin{aligned}
\mathrm{dc}\,x &= 1 + \Sigma_1^\infty h'\psi_{n-1}(h', 1)\,x_{2n},\\
\mathrm{nc}\,x &= 1 + \Sigma_1^\infty \psi_{n-1}(1, h')\,x_{2n},\\
\mathrm{sc}\,x &= \quad \Sigma_1^\infty \phi_{n-1}(1, h')\,x_{2n-1},\\
\mathrm{zc}\,x &= -\mathrm{e}\,x - \Sigma_1^\infty h' f_{n-1}(1, h')\,x_{2n+1};
\end{aligned}$$

$$\begin{aligned}
\mathrm{cd}\,x &= 1 - \Sigma_1^\infty (-)^{n-1} h'\psi_{n-1}(h', -h)\,x_{2n},\\
\mathrm{sd}\,x &= \quad \Sigma_1^\infty \phi_{n-1}(h, -h')\,x_{2n-1},\\
\mathrm{nd}\,x &= 1 + \Sigma_1^\infty k^2\psi_{n-1}(h, -h')\,x_{2n},\\
\mathrm{zd}\,x &= -\mathrm{g}\,x + \Sigma_1^\infty hh' f_{n-1}(h, h')\,x_{2n+1};
\end{aligned}$$

$$\begin{aligned}
\mathrm{sn}\,x &= \quad \Sigma_1^\infty (-)^{n-1} \phi_{n-1}(1, h)\,x_{2n-1},\\
\mathrm{cn}\,x &= 1 - \Sigma_1^\infty (-)^{n-1} \psi_{n-1}(1, h)\,x_{2n},\\
\mathrm{dn}\,x &= 1 - \Sigma_1^\infty (-)^{n-1} \psi_{n-1}(h, 1)\,x_{2n},\\
\mathrm{zn}\,x &= -\mathrm{i}\,x - \Sigma_1^\infty (-)^{n-1} h f_{n-1}(1, h)\,x_{2n+1}.
\end{aligned}$$

§ 28. If these equations be arranged in classes depending upon the form of the coefficients, the twelve elliptic functions separate themselves into groups of three, three, and six, viz. :

$$\begin{aligned}
\mathrm{ns}\,x &= x^{-1} + \Sigma_1^\infty \chi_n(1, h)\,x_{2n-1},\\
\mathrm{ds}\,x &= x^{-1} + \Sigma_1^\infty \chi_n(h', -h)\,x_{2n-1},\\
\mathrm{cs}\,x &= x^{-1} + \Sigma_1^\infty (-)^n \chi_n(1, h')\,x_{2n-1};
\end{aligned}$$

$$\begin{aligned}
\mathrm{sc}\,x &= \Sigma_1^\infty \phi_{n-1}(1, h')\,x_{2n-1},\\
\mathrm{sd}\,x &= \Sigma_1^\infty \phi_{n-1}(h, -h')\,x_{2n-1},\\
\mathrm{sn}\,x &= \Sigma_1^\infty (-)^{n-1} \phi_{n-1}(1, h)\,x_{2n-1};
\end{aligned}$$

$$\begin{aligned}
\mathrm{cn}\,x &= 1 + \Sigma_1^\infty (-)^n \psi_{n-1}(1, h)\,x_{2n},\\
\mathrm{dn}\,x &= 1 + \Sigma_1^\infty (-)^n h\psi_{n-1}(h, 1)\,x_{2n},\\
\mathrm{nc}\,x &= 1 + \Sigma_1^\infty \psi_{n-1}(1, h')\,x_{2n},\\
\mathrm{dc}\,x &= 1 + \Sigma_1^\infty h'\psi_{n-1}(h', 1)\,x_{2n},\\
\mathrm{nd}\,x &= 1 + \Sigma_1^\infty h\psi_{n-1}(h, -h')\,x_{2n},\\
\mathrm{cd}\,x &= 1 + \Sigma_1^\infty (-)^n h'\psi_{n-1}(h', -h)\,x_{2n}.
\end{aligned}$$

Thus the three functions having s as their final letter form one group, the three having s as their initial letter form another group, and the six even functions form a third group.

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Similarly the four Zeta functions separate themselves into groups of one and three, the former consisting of the one uneven function and the latter of the three even functions, viz.:

$$\mathrm{zs}\,x = x^{-1} - \tfrac{1}{3}(\mathrm{i}+\mathrm{g}+\mathrm{e})\,x - \Sigma_1^\infty F_{n+1}(1, h)\,x_{2n+1};$$

$$\mathrm{zc}\,x = -\mathrm{e}\,x - \Sigma_1^\infty h' f_{n-1}(1, h')\,x_{2n+1},$$

$$\mathrm{zd}\,x = -\mathrm{g}\,x + \Sigma_1^\infty hh' f_{n-1}(h, -h')\,x_{2n+1},$$

$$\mathrm{zn}\,x = -\mathrm{i}\,x + \Sigma_1^\infty (-)^n h f_{n-1}(1, h)\,x_{2n+1}.$$

The five coefficients ϕ_n, ψ_n, χ_n, f_n, F_n, § 29.

§ 29. The five coefficients $\phi_n(a, b)$, &c. are all homogeneous functions of a and b of the order n. Since the suffix attached to the functional sign always indicates the dimensions of the function, it follows that $\phi_n(-a, -b) = (-)^n \phi_n(a, b)$, &c.

The two arguments are involved symmetrically in each of the four functions ϕ_n, χ_n, f_n, F_n; so that

$$\phi_n(a, b) = \phi_n(b, a), \qquad \chi_n(a, b) = \phi_n(b, a),$$
$$f_n(a, b) = f_n(b, a), \qquad F_n(a, b) = F_n(b, a);$$

but this is not the case with the function $\psi_n(a, b)$, which is not equal to $\psi(b, a)$.

The function $F_n(a, b)$ possesses the additional property of being unaltered by the substitution of $a - b$ and $-b$ for a and b respectively; so that

$$F_n(a, b) = F_n(b, a) = F_n(a - b, -b) = F_n(b - a, -a).$$

For example,

$$F_n(1, h) = (-)^n F_n(1, h') = F_n(h', -h).$$

Values of $\phi_n(a, b)$, $\psi_n(a, b)$, &c. for $n = 1, 2, \ldots 5$, § 30.

§ 30. The values* of the coefficients $\phi_n(a, b)$, $\psi_n(a, b)$, &c. as far as $n = 5$ are given on the next page. The notation

$$(p_0, p_1, p_2, \ldots, p_n)\,(a, b)^n$$

is used to represent the expression

$$p_0 a^n + p_1 a^{n-1} b + p_2 a^{n-2} b^2 + \ldots + p_n b^n.$$

* These values are taken from § 381 (p. 335) of my *Lectures on Elliptic Functions*. This work is not yet published.

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$$\phi_0(a, b) = 1,$$
$$\phi_1(a, b) = (1, 1)(a, b),$$
$$\phi_2(a, b) = (1, 14, 1)(a, b)^2,$$
$$\phi_3(a, b) = (1, 135, 135, 1)(a, b)^3,$$
$$\phi_4(a, b) = (1, 1228, 5478, 1228, 1)(a, b)^4,$$
$$\phi_5(a, b) = (1, 11069, 165826, 165826, 11069, 1)(a, b)^5;$$

$$\psi_0(a, b) = 1,$$
$$\psi_1(a, b) = (1, 4)(a, b),$$
$$\psi_2(a, b) = (1, 44, 16)(a, b)^2,$$
$$\psi_3(a, b) = (1, 408, 912, 64)(a, b)^3,$$
$$\psi_4(a, b) = (1, 3688, 30768, 15808, 256)(a, b)^4,$$
$$\psi_5(a, b) = (1, 33212, 870640, 1538560, 259328, 1024)(a, b)^5;$$

$$\chi_1(a, b) = \tfrac{1}{6}(1, 1)(a, b),$$
$$\chi_2(a, b) = \tfrac{1}{60}(7, -22, 7)(a, b)^2,$$
$$\chi_3(a, b) = \tfrac{1}{1260}(31, -15, -15, 31)(a, b)^3,$$
$$\chi_4(a, b) = \tfrac{1}{3360}(381, -852, 558, -852, 381)(a, b)^4,$$
$$\chi_5(a, b) = \tfrac{1}{6720}(511, -1261, 1006, 1006, -1261, 511)(a, b)^5;$$

$$f_0(a, b) = 2,$$
$$f_1(a, b) = 2^3(1, 1)(a, b),$$
$$f_2(a, b) = 2^4(2, 13, 2)(a, b)^2,$$
$$f_3(a, b) = 2^7(1, 30, 30, 1)(a, b)^3,$$
$$f_4(a, b) = 2^8(2, 251, 876, 251, 2)(a, b)^4,$$
$$f_5(a, b) = 2^{11}(1, 509, 4951, 4951, 509, 1)(a, b)^5;$$

$$F_1(a, b) = \tfrac{1}{3}(1, 1)(a, b),$$
$$F_2(a, b) = \tfrac{2}{15}(1, -1, 1)(a, b)^2,$$
$$F_3(a, b) = \tfrac{8}{63}(2, -3, -3, 2)(a, b)^3,$$
$$F_4(a, b) = \tfrac{16}{105}(1, -2, 3, -2, 1)(a, b)^4,$$
$$F_5(a, b) = \tfrac{128}{693}(2, -5, 2, 2, -5, 2)(a, b)^5.$$

I have calculated the values of $F_n(a, b)$ as far as $n = 11$. They will be given in a subsequent Chapter.

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Values of the coefficients for the arguments 0 *and* 1, § 31.

§ 31. The general values of the quantities $\phi_n(a, b)$, &c., can be easily assigned in the cases in which the arguments are (0, 1), (1, 0), or (1, 1). For, denoting by B_n and E_n the n^{th} Bernoullian and Eulerian numbers, we have

$$\sin x = \quad \Sigma_1^\infty (-)^{n-1} x_{2n-1},$$

$$\cos x = 1 + \Sigma_1^\infty (-)^n x_{2n},$$

$$\tan x = \quad \Sigma_1^\infty 2^{2n-1}(2^{2n}-1)\frac{B_n}{n} x_{2n-1},$$

$$\sec x = 1 + \Sigma_1^\infty E_n x_{2n},$$

$$\cot x = x^{-1} - \Sigma_1^\infty 2^{2n-1}\frac{B_n}{n} x_{2n-1},$$

$$\operatorname{cec} x = x^{-1} + \Sigma_1^\infty (2^{2n-1}-1)\frac{B_n}{n} x_{2n-1};$$

and also

$$\sin^2 x = \quad \Sigma_1^\infty (-)^{n-1} 2^{2n-1} x_{2n},$$

$$\tan^2 x = \quad \Sigma_1^\infty 2^{2n+1}(2^{2n+2}-1)\frac{B_{n+1}}{n+1} x_{2n},$$

$$\cot^2 x = x^{-2} - \tfrac{2}{3} + \Sigma_1^\infty 2^{2n+1}\frac{B_{n+1}}{n+1} x_{2n}.$$

We thus find

$$\phi_{n-1}(1, 0) = 1, \qquad \phi_{n-1}(1, 1) = 2^{2n-1}(2^{2n}-1)\frac{B_n}{n},$$

$$\psi_{n-1}(1, 0) = 1, \qquad \psi_{n-1}(1, 1) = E_n,$$

$$\chi_n(1, 0) = (2^{n-1}-1)\frac{B_n}{n}, \quad \chi_n(1, 1) = (-)^{n-1}2^{2n-1}\frac{B_n}{n},$$

$$f_{n-2}(1, 0) = 2^{2n-3}, \qquad f_{n-2}(1, 1) = 2^{2n-1}(2^{2n}-1)\frac{B_n}{n},$$

$$F_n(1, 0) = (-)^n F_n(1, 1) = 2^{2n-1}\frac{B_n}{n},$$

and $$\psi_n(0, 1) = 2^{2n}\psi_n(1, 0) = 2^{2n}.$$

It will be remembered (§ 29) that the four functions ϕ_n, χ_n, f_n, F_n are unaltered by the interchange of the two arguments, so that $\phi_n(0, 1) = \phi_n(1, 0)$, &c.

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Approximate expressions for the elliptic and Zeta functions in powers of q and k, §§ 32–35.

§ 32. In §§ 7–12 the sixteen elliptic and Zeta functions $\rho\,\mathrm{ns}\,\rho x$, &c. have been expressed as series proceeding by powers of q, the coefficients of the general terms consisting of groups of sines and cosines depending upon the divisors of the exponent of q or of its half or double.

As it is convenient for many purposes to have for reference the first few terms of the expansions of the elliptic and Zeta functions themselves (viz. without the factors ρ, $k\rho$, &c.), I here give their values, as also those of $\mathrm{s}(\rho x)$, &c., as far as the term involving q^2. As usual the letter u denotes ρx.

$$\mathrm{s}(u) = \sin x \{1 + 4q + 4q^2(1 + \sin^2 x)\},$$
$$\mathrm{c}(u) = \cos x \{1 - 4q^2 \sin^2 x\},$$
$$\mathrm{d}(u) = 1 - 4q \sin^2 x + 8q^2 \sin^2 x,$$
$$\mathrm{n}(u) = 1 + 4q \sin^2 x + 8q^2 \sin^2 x;$$

$$\mathrm{ns}\,u = \mathrm{cec}\,x \{1 - 4q \cos^2 x + 12q^2 \cos^2 x\},$$
$$\mathrm{ds}\,u = \mathrm{cec}\,x \{1 - 4q(1 + \sin^2 x) + 4q^2(3 + 5 \sin^2 x)\},$$
$$\mathrm{cs}\,u = \cot x \{1 - 4q + 4q^2(3 - 2 \sin^2 x)\},$$
$$\mathrm{zs}\,u = \cot x \{1 - 4q + 4q^2(3 + 2 \sin^2 x)\};$$

$$\mathrm{dc}\,u = \sec x \{1 - 4q \sin^2 x + 12q^2 \sin^2 x\},$$
$$\mathrm{nc}\,u = \sec x \{1 + 4q \sin^2 x + 12q^2 \sin^2 x\},$$
$$\mathrm{sc}\,u = \tan x \{1 + 4q + 4q^2(3 - 2 \cos^2 x)\},$$
$$\mathrm{zc}\,u = -\tan x \{1 - 4q + 4q^2(3 + 2 \cos^2 x)\};$$

$$\mathrm{cd}\,u = \cos x \{1 + 4q \sin^2 x - 4q^2 \sin x \sin 3x\},$$
$$\mathrm{sd}\,u = \sin x \{1 + 4q(1 + \sin^2 x) + 4q^2(1 + 3 \sin^2 x + 4 \sin^4 x)\},$$
$$\mathrm{nd}\,u = 1 + 8q \sin^2 x + 32q^2 \sin^4 x,$$
$$\mathrm{zd}\,u = -8 \sin x \cos x \{q - 2q^2(1 + 2 \cos^2 x)\};$$

$$\mathrm{sn}\,u = \sin x \{1 + 4q \cos^2 x + 4q^2 \cos x \cos 3x\},$$
$$\mathrm{cn}\,u = \cos x \{1 - 4q \sin^2 x - 4q^2 \sin x \sin 3x\},$$
$$\mathrm{dn}\,u = 1 - 8q \sin^2 x + 32q^2 \sin^4 x,$$
$$\mathrm{zn}\,u = 8 \sin x \cos x \{q - 2q^2(1 + 2 \sin^2 x)\}.$$

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The preceding expressions of s (u), &c. may be obtained from their values as quotients of q-series; and the expansions of the sixteen elliptic and Zeta functions may be readily deduced from those of s (u), &c. by division. The latter expansions may also be derived from the system in § 8 by multiplying by the q-series for ρ^{-1}, &c., the values of which, as far as the term involving $q^{\frac{1}{2}}$, are:

$$\rho^{-1} = 1 - 4q + 12q^2,^*$$
$$(k'\rho)^{-1} = 1 + 4q + 12q^2,$$
$$(k\rho)^{-1} = \tfrac{1}{4}q^{-\frac{1}{2}}(1 - 2q^{\frac{1}{2}}),$$
$$(kk'\rho)^{-1} = \tfrac{1}{4}q^{-\frac{1}{2}}(1 + 8q + 30q^{\frac{3}{2}}).$$

§ 33. To pass from q to h we have the formulæ:

$$q = e^{-\mu}, \quad \mu = \frac{\pi K'}{K}, \quad K' = K_1 \log\frac{4}{k} - K_2,$$

$$K_1 = \frac{2K}{\pi} = 1 + \Sigma_1^\infty P_n h^n,$$

$$K_2 = 2\Sigma_1^\infty P_n \left(1 - \tfrac{1}{2} + \tfrac{1}{3} - \ldots - \frac{1}{2n}\right) h^n,$$

and
$$P_n = \frac{1^2 . 3^2 . 5^2 \ldots (2n-1)^2}{2^2 . 4^2 . 6^2 \ldots (2n)^2}.$$

Thus we find

$$\mu = \log\frac{16}{h} - 2\frac{K_2}{K_1},$$

so that
$$q = \tfrac{1}{16}he^{4\gamma}$$

---

* The expansion of ρ^{-1} as far as q^{27} is given in a paper in vol. XXI. of the *Quarterly Journal* (p. 66). Let $4g_n$ denote the coefficient of $(-q)^n$, so that

$$\frac{1}{\rho} = 1 - 4g_1 q + 4g_2 q^2 - 4g_3 q^3 + \text{&c.}, \text{ and } \frac{1}{k'\rho} = 1 + 4g_1 q + 4g_2 q^2 + 4g_3 q^3 + \text{&c.},$$

then

$g_1 = 1$, $g_2 = 3$, $g_3 = 8$, $g_4 = 19$, $g_5 = 42$, $g_6 = 88$, $g_7 = 176$, $g_8 = 339$,
$g_9 = 638$, $g_{10} = 1150$, $g_{11} = 2040$, $g_{12} = 3544$, $g_{13} = 6042$, $g_{14} = 10128$,
$g_{15} = 16720$, $g_{16} = 27219$, $g_{17} = 43746$, $g_{18} = 69483$, $g_{19} = 109160$,
$g_{20} = 169758$, $g_{21} = 261504$, $g_{22} = 399272$, $g_{23} = 604560$,
$g_{24} = 908248$, $g_{25} = 1354427$, $g_{26} = 2005710$, $g_{27} = 2950544$.

Various formulæ for the calculation of g_n are also considered in the paper.

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where γ denotes the quotient

$$\frac{\frac{1}{2^2}(1-\frac{1}{2})h+\frac{1^2.3^2}{2^2.4^2}(1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4})h^2+\frac{1^2.3^2.5^2}{2^2.4^2.6^2}(1-\frac{1}{2}+\ldots-\frac{1}{6})h^3+\&c.}{1+\frac{1}{2^2}h+\frac{1^2.3^2}{2^2.4^2}h^2+\frac{1^2.3^2.5^2}{2^2.4^2.6^2}h^3+\&c.}.$$

The expansion of q in ascending powers of h is thus easily found to be

$$q=\tfrac{1}{16}h+\tfrac{1}{32}h^2+\tfrac{21}{1024}h^3+\&c.$$

§ 34. By substituting $\frac{1}{16}h+\frac{1}{32}h^2$ for q in the formulæ of § 32 we obtain the following expansions of $s(\rho x)$, &c., and of the elliptic and Zeta functions as far as the term involving h^2:

$$s(u)=\sin x\{1+\tfrac{1}{4}h+\tfrac{1}{64}h^2(9+\sin^2 x)\},$$
$$c(u)=\cos x\{1-\tfrac{1}{64}h^2\sin^2 x\},$$
$$d(u)=1-\tfrac{1}{4}h\sin^2 x-\tfrac{3}{32}h^2\sin^2 x,$$
$$n(u)=1+\tfrac{1}{4}h\sin^2 x+\tfrac{5}{32}h^2\sin^2 x.$$

$$ns\,u=\operatorname{cec} x\{1-\tfrac{1}{4}h\cos^2 x-\tfrac{5}{64}h^2\cos^2 x\},$$
$$ds\,u=\operatorname{cec} x\{1-\tfrac{1}{4}h(1+\sin^2 x)-\tfrac{1}{64}h^2(5+3\sin^2 x)\},$$
$$cs\,u=\cot x\{1-\tfrac{1}{4}h-\tfrac{1}{64}h^2(5+2\sin^2 x)\},$$
$$zs\,u=\cot x\{1-\tfrac{1}{4}h-\tfrac{1}{64}h^2(5-2\sin^2 x)\};$$

$$dc\,u=\quad\sec x\{1-\tfrac{1}{4}h\sin^2 x\quad\tfrac{5}{64}h^2\sin^2 x\},$$
$$nc\,u=\quad\sec x\{1+\tfrac{1}{4}h\sin^2 x+\tfrac{1}{64}h^2\sin^2 x\},$$
$$sc\,u=\quad\tan x\{1+\tfrac{1}{4}h+\tfrac{1}{64}h^2(9+2\sin^2 x)\},$$
$$zc\,u=-\tan x\{1-\tfrac{1}{4}h-\tfrac{1}{64}h^2(3+2\sin^2 x)\};$$

$$cd\,u=\quad\cos x\{1+\tfrac{1}{4}h\sin^2 x+\tfrac{1}{64}h^2(5\sin^2 x+4\sin^4 x)\},$$
$$sd\,u=\quad\sin x\{1+\tfrac{1}{4}h(1+\sin^2 x)+\tfrac{1}{64}h^2(9+11\sin^2 x+4\sin^4 x)\},$$
$$nd\,u=\quad 1+\tfrac{1}{2}h\sin^2 x+\tfrac{1}{8}h^2(2\sin^2 x+\sin^4 x),$$
$$zd\,u=-\tfrac{1}{2}\sin x\cos x\{h+\tfrac{1}{8}h^2(1+2\sin^2 x)\};$$

$$sn\,u=\sin x\{1+\tfrac{1}{4}h\cos^2 x+\tfrac{1}{64}h^2(5\cos^2 x+4\cos^4 x)\},$$
$$cn\,u=\cos x\{1-\tfrac{1}{4}h\sin^2 x-\tfrac{1}{64}h^2\sin^2 x(7+4\cos^2 x)\},$$
$$dn\,u=1-\tfrac{1}{2}h\sin^2 x-\tfrac{1}{8}h^2\sin^2 x(1+\cos^2 x),$$
$$zn\,u=\tfrac{1}{2}\sin x\cos x\{h+\tfrac{1}{8}h^2(1+2\cos^2 x)\};$$

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§ 35. In the expansions contained in the last section and in § 32 the argument of the Theta, elliptic and Zeta functions is u or ρx. We can deduce the corresponding expansions for the argument x by replacing x by $\frac{x}{\rho}$ in the right-hand members of the equations and expanding each term in powers of q.

Taking for example the values of $\mathrm{sn}\,u$, $\mathrm{cn}\,u$, $\mathrm{dn}\,u$, we find, by replacing x by $(1-4q+12q^2)\,x$ and expanding in powers of q,

$$\mathrm{sn}\,x = \sin x - 4q\cos x\,(x - \sin x\cos x)$$
$$- 4q^2\{2x^2\sin x + x\cos x\,(1 - 12\sin^2 x) - \sin x\cos x\cos 3x\},$$
$$\mathrm{cn}\,x = \cos x + 4q\sin x\,(x - \sin x\cos x)$$
$$- 4q^2\{2x^2\cos x - x\sin x\,(5 - 12\sin^2 x) + \sin x\cos x\sin 3x\},$$
$$\mathrm{dn}\,x = 1 - 8q\sin^2 x + 32q^2\,(2x\sin x\cos x + \sin^4 x).$$

By substituting $\frac{1}{16}h + \frac{1}{8}\frac{1}{2}h^2$ for q we obtain, as in the last section, the corresponding expansions in powers of h, viz.

$$\mathrm{sn}\,x = \sin x - \tfrac{1}{4}h\cos x\,(x - \sin x\cos x)$$
$$- \tfrac{1}{8}\tfrac{1}{4}h^2\{2x^2\sin x - 3x\cos x\,(1 - 4\cos^2 x) - \sin x\cos^2 x\,(5 + 4\cos^2 x)\},$$
$$\mathrm{cn}\,x = \cos x + \tfrac{1}{4}h\sin x\,(x - \sin x\cos x)$$
$$- \tfrac{1}{8}\tfrac{1}{4}h^2\{2x^2\cos x - x\sin x\,(1 + 12\cos^2 x) + \sin^2 x\cos x\,(7 + 4\cos^2 x)\},$$
$$\mathrm{dn}\,x = 1 - \tfrac{1}{2}h\sin^2 x + \tfrac{1}{8}h^2\{2x\sin x\cos x - \sin^2 x\,(1 + \cos^2 x)\}.$$

On the derivation of trigonometrical identities from results in Elliptic Functions, § 36.

§ 36. In connexion with the formulæ in §§ 32 and 34 it may be remarked that an interesting application of these expansions occurs in the derivation of trigonometrical identities from results in Elliptic Functions. Of course in any formulæ in Elliptic Functions we may put $k = 0$, and replace $\mathrm{sn}\,x$, $\mathrm{cn}\,x$ and $\mathrm{dn}\,x$ by $\sin x$, $\cos x$ and unity respectively, but it frequently happens that by putting $k = 0$ we cause the disappearance of some of the most characteristic terms in the expression. If, however, we suppose the arguments to be multiplied

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throughout by ρ and equate the coefficients of q or h in the expanded expressions, we often obtain trigonometrical formulæ of greater interest. As the quantities h or h' may be involved in the coefficients it is generally more convenient to expand in powers of h.

For example, taking the elliptic-function formula

$$h' - hh' \operatorname{sn}\alpha \operatorname{sn}\beta \operatorname{sn}\gamma \operatorname{sn}\delta + h \operatorname{cn}\alpha \operatorname{cn}\beta \operatorname{cn}\gamma \operatorname{cn}\delta - \operatorname{dn}\alpha \operatorname{dn}\beta \operatorname{dn}\gamma \operatorname{dn}\delta = 0,$$

in which α, β, γ, δ are any quantities whose sum is zero, and using the formulæ

$$\operatorname{sn}\rho x = \sin x (1 + \tfrac{1}{4}h \cos^2 x),$$

$$\operatorname{cn}\rho x = \cos x (1 - \tfrac{1}{4}h \sin^2 x),$$

$$\operatorname{dn}\rho x = 1 - \tfrac{1}{2}h \sin^2 x,$$

we find

$$\cos\alpha \cos\beta \cos\gamma \cos\delta - \sin\alpha \sin\beta \sin\gamma \sin\delta = 1 - \tfrac{1}{2}(\sin^2\alpha + \sin^2\beta + \sin^2\gamma + \sin^2\delta).$$

Similarly, from the elliptic-function identities

$$s_1 s_2 + s_3 s_4 + s_5 s_6 + h s_1 s_2 s_3 s_4 s_5 s_6 = 0,$$

$$s_2 s_3 s_5 + s_4 s_5 s_1 + s_6 s_1 s_3 + s_2 s_4 s_6 = 0,$$

in which s denotes sn, and the suffixes 1, 2, ... 6 refer to the six arguments a, $b-c$, b, $c-a$, c, $a-b$, respectively, a, b, c being unrestricted, we find, by putting $h=0$,

$$\Sigma \sin a \sin(b-c) = 0,$$

$$\Sigma \sin b \sin c \sin(b-c) + \sin(b-c) \sin(c-a) \sin(a-b) = 0,$$

and, by equating the coefficients of h,

$$\Sigma \{\sin^2 a \sin(b-c) + \sin a \sin^2(b-c)\} = 4 \sin a \sin b \sin c \sin(b-c) \sin(c-a) \sin(a-b),$$

$$\Sigma \sin b \sin c \sin(b-c) \{\sin^2 b + \sin^2 c + \sin^2(b-c)\} + \sin(b-c) \sin(c-a) \sin(a-b) \times \{\sin^2(b-c) + \sin^2(c-a) + \sin^2(a-b)\} = 0.*$$

* The above mode of deriving trigonometrical identities was considered in vol. x., pp. 92-97.

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DOUBLY INFINITE SERIES.

All the developments given in the preceding sections have been singly infinite. I now proceed to consider the various forms of the doubly infinite series by which the elliptic and Zeta functions may be represented.

Series proceeding by powers of q and $e^{\pi x}$, §§ 37–40.

§ 37. By expanding in ascending powers of q the coefficients in the original form of the q-series (vol. XVII., pp. 2, 3) we obtain the following doubly infinite series for the sixteen functions.

The summations refer to the letters r and s which are to have all positive integral values from 1 to ∞. For the sake of greater regularity in writing the formulæ, the symbols r_1, r_2, s_1, s_2 are used with the meanings:

$$r_1 = 2r - 1, \qquad r_2 = 2r,$$
$$s_1 = 2s - 1, \qquad s_2 = 2s.$$

$$\rho\,\mathrm{ns}\,\rho x = \mathrm{cec}\,x + 4\Sigma_1^\infty \Sigma_1^\infty q^{\frac{1}{2}r_2 s_1} \sin s_1 x,$$
$$\rho\,\mathrm{ds}\,\rho x = \mathrm{cec}\,x + 4\Sigma_1^\infty \Sigma_1^\infty (-)^r q^{\frac{1}{2}r_2 s_1} \sin s_1 x,$$
$$\rho\,\mathrm{cs}\,\rho x = \cot x + 4\Sigma_1^\infty \Sigma_1^\infty (-)^r q^{\frac{1}{2}r_2 s_2} \sin s_2 x,$$
$$\rho\,\mathrm{zs}\,\rho x = \cot x + 4\Sigma_1^\infty \Sigma_1^\infty q^{\frac{1}{2}r_2 s_2} \sin s_2 x;$$

$$\rho\,\mathrm{dc}\,\rho x = \sec x - 4\Sigma_1^\infty \Sigma_1^\infty (-)^s q^{\frac{1}{2}r_2 s_1} \cos s_1 x,$$
$$k'\rho\,\mathrm{nc}\,\rho x = \sec x - 4\Sigma_1^\infty \Sigma_1^\infty (-)^{r+s} q^{\frac{1}{2}r_2 s_1} \cos s_1 x,$$
$$k'\rho\,\mathrm{sc}\,\rho x = \tan x - 4\Sigma_1^\infty \Sigma_1^\infty (-)^{r+s} q^{\frac{1}{2}r_2 s_2} \sin s_2 x,$$
$$\rho\,\mathrm{zc}\,\rho x = -\tan x + 4\Sigma_1^\infty \Sigma_1^\infty (-)^s q^{\frac{1}{2}r_2 s_2} \sin s_2 x;$$

$$k\rho\,\mathrm{cd}\,\rho x = -4\Sigma_1^\infty \Sigma_1^\infty (-)^s q^{\frac{1}{2}r_1 s_1} \cos s_1 x,$$
$$kk'\rho\,\mathrm{sd}\,\rho x = 4\Sigma_1^\infty \Sigma_1^\infty (-)^{r+s} q^{\frac{1}{2}r_1 s_1} \sin s_1 x,$$
$$k'\rho\,\mathrm{nd}\,\rho x = 1 - 4\Sigma_1^\infty \Sigma_1^\infty (-)^{r+s} q^{\frac{1}{2}r_1 s_2} \cos s_2 x,$$
$$\rho\,\mathrm{zd}\,\rho x = 4\Sigma_1^\infty \Sigma_1^\infty (-)^s q^{\frac{1}{2}r_1 s_2} \sin s_2 x;$$

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$$k\rho\ \mathrm{sn}\,\rho x = 4\Sigma_1^\infty \Sigma_1^\infty q^{\frac{1}{2}r_1 s_1} \sin s_1 x,$$

$$k\rho\ \mathrm{cn}\,\rho x = -4\Sigma_1^\infty \Sigma_1^\infty (-)^r q^{\frac{1}{2}r_1 s_1} \cos s_1 x,$$

$$\rho\ \mathrm{dn}\,\rho x = 1 - 4\Sigma_1^\infty \Sigma_1^\infty (-)^r q^{\frac{1}{2}r_1 s_2} \cos s_2 x,$$

$$\rho\ \mathrm{zn}\,\rho x = 4\Sigma_1^\infty \Sigma_1^\infty q^{\frac{1}{2}r_1 s_2} \sin s_2 x.$$

In the preceding formulæ we may regard r_1 and s_1 as denoting any two even numbers and r_2 and s_2 as denoting any two even numbers. Thus r_1 and s_1 are to have all positive integral uneven values from 1 to ∞ and r_2 and s_2 all positive integral even values from 2 to ∞. The sign $(-)^r$ will then denote $(-)^{\frac{1}{2}(r_1+1)}$ or $(-)^{\frac{1}{2}r_2}$ according as r_1 or r_2 occurs in the series; and, similarly $(-)^s$ will denote $(-)^{\frac{1}{2}(s_1+1)}$ or $(-)^{\frac{1}{2}s_2}$.

§ 38. We may express the preceding double series in a form in which the limits are $\pm\infty$ if we make the convention that the exponent of q is always to be positive, and that when it has the negative sign this sign is to be transformed to the whole coefficient, viz. *if we interpret* q^{-n} *to mean* $-q^n$.*

Adopting this convention, and putting z for e^{ix}, we obtain the following formulæ, which are divided into groups according to the form of the series:

$$k\rho\ \mathrm{sn}\,\rho x = -i\Sigma_{-\infty}^\infty \Sigma_{-\infty}^\infty q^{\frac{1}{2}r_1 s_1} z^{s_1},$$

$$k\rho\ \mathrm{cn}\,\rho x = -i\Sigma_{-\infty}^\infty \Sigma_{-\infty}^\infty i^{r_1} q^{\frac{1}{2}r_1 s_1} z^{s_1},$$

$$k\rho\ \mathrm{cd}\,\rho x = -i\Sigma_{-\infty}^\infty \Sigma_{-\infty}^\infty i^{s_1} q^{\frac{1}{2}r_1 s_1} z^{s_1},$$

$$kk'\rho\ \mathrm{sd}\,\rho x = i\Sigma_{-\infty}^\infty \Sigma_{-\infty}^\infty i^{r_1+s_1} q^{\frac{1}{2}r_1 s_1} z^{s_1};$$

$$\rho\ \mathrm{zn}\,\rho x = -i\Sigma_{-\infty}^\infty \Sigma_{-\infty}^\infty q^{\frac{1}{2}r_1 s_2} z^{s_2},$$

$$\rho\ \mathrm{dn}\,\rho x = -i\Sigma_{-\infty}^\infty \Sigma_{-\infty}^\infty i^{r_1} q^{\frac{1}{2}r_1 s_2} z^{s_2},$$

$$\rho\ \mathrm{zd}\,\rho x = -i\Sigma_{-\infty}^\infty \Sigma_{-\infty}^\infty i^{s_2} q^{\frac{1}{2}r_1 s_2} z^{s_2},$$

$$k'\rho\ \mathrm{nd}\,\rho x = -i\Sigma_{-\infty}^\infty \Sigma_{-\infty}^\infty i^{r_1+s_2} q^{\frac{1}{2}r_1 s_2} z^{s_2};$$

* We may, for example, represent the value to be assigned to q^n by $\frac{n}{\sqrt{n^2}}\, q^{\sqrt{n^2}}$, the positive sign being always assigned to the radical.

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$$\rho\ \mathrm{ns}\rho x = \mathrm{cec}\,x - i\,\Sigma_{-\infty}^{\infty}\Sigma_{-\infty}^{\infty} q^{\frac{1}{2}r_2s_1} z^{s_1},$$

$$\rho\ \mathrm{ds}\rho x = \mathrm{cec}\,x - i\,\Sigma_{-\infty}^{\infty}\Sigma_{-\infty}^{\infty} i^{r_2} q^{\frac{1}{2}r_2s_1} z^{s_1},$$

$$\rho\ \mathrm{dc}\rho x = \mathrm{sec}\,x - i\,\Sigma_{-\infty}^{\infty}\Sigma_{-\infty}^{\infty} i^{s_1} q^{\frac{1}{2}r_2s_1} z^{s_1},$$

$$k'\rho\ \mathrm{nc}\rho x = \mathrm{sec}\,x - i\,\Sigma_{-\infty}^{\infty}\Sigma_{-\infty}^{\infty} i^{r_2+s_1} q^{\frac{1}{2}r_2s_1} z^{s_1};$$

$$\rho\ \mathrm{zs}\rho x = \cot x - i\,\Sigma_{-\infty}^{\infty}\Sigma_{-\infty}^{\infty} q^{\frac{1}{2}r_2s_2} z^{s_2},$$

$$\rho\ \mathrm{cs}\rho x = \cot x - i\,\Sigma_{-\infty}^{\infty}\Sigma_{-\infty}^{\infty} i^{r_2} q^{\frac{1}{2}r_2s_2} z^{s_2},$$

$$\rho\ \mathrm{zc}\rho x = -\tan x - i\,\Sigma_{-\infty}^{\infty}\Sigma_{-\infty}^{\infty} i^{s_2} q^{\frac{1}{2}r_2s_2} z^{s_2},$$

$$k'\rho\ \mathrm{sc}\rho x = \tan x + i\,\Sigma_{-\infty}^{\infty}\Sigma_{-\infty}^{\infty} i^{r_2+s_2} q^{\frac{1}{2}r_2s_2} z^{s_2}.$$

In the above equations the groups are the same as in §9, but the order of the members is different in the first two groups.*

§ 39. The powers of i have been introduced for the sake of greater regularity in writing the formulæ. They may be removed at sight by replacing i^{r_1} by $-(-)^r i$ and i^{r_2} by $(-)^r$ and making similar changes in the case of s. Thus, for example,

$$k\rho\ \mathrm{sn}\rho x = -i\,\Sigma_{-\infty}^{\infty}\Sigma_{-\infty}^{\infty} q^{\frac{1}{2}r_1s_1} z^{s_1},$$

$$k\rho\ \mathrm{cn}\rho x = -\Sigma_{-\infty}^{\infty}\Sigma_{-\infty}^{\infty} (-)^r q^{\frac{1}{2}r_1s_1} z^{s_1},$$

$$k\rho\ \mathrm{cd}\rho x = -\Sigma_{-\infty}^{\infty}\Sigma_{-\infty}^{\infty} (-)^s q^{\frac{1}{2}r_1s_1} z^{s_1},$$

$$kk'\rho\ \mathrm{sd}\rho x = -i\,\Sigma_{-\infty}^{\infty}\Sigma_{-\infty}^{\infty} (-)^{r+s} q^{\frac{1}{2}r_1s_1} z^{s_1};$$

$$\rho\ \mathrm{zn}\rho x = -i\,\Sigma_{-\infty}^{\infty}\Sigma_{-\infty}^{\infty} q^{\frac{1}{2}r_1s_2} z^{s_2},$$

$$\rho\ \mathrm{dn}\rho x = -\Sigma_{-\infty}^{\infty}\Sigma_{-\infty}^{\infty} (-)^r q^{\frac{1}{2}r_1s_2} z^{s_2},$$

$$\rho\ \mathrm{zd}\rho x = -i\,\Sigma_{-\infty}^{\infty}\Sigma_{-\infty}^{\infty} (-)^s q^{\frac{1}{2}r_1s_2} z^{s_2},$$

$$k'\rho\ \mathrm{nd}\rho x = -\Sigma_{-\infty}^{\infty}\Sigma_{-\infty}^{\infty} (-)^{r+s} q^{\frac{1}{2}r_1s_2} z^{s_2},$$

&c. &c.

* It will be noticed that the exponents $\frac{1}{2}m$, n, n, $2n$ which occur in the groups in §§ 7–14 are of the forms $\frac{1}{2}r_1s_1$, $\frac{1}{2}r_1s_2$, $\frac{1}{2}r_2s_1$, $\frac{1}{2}r_2s_2$, *i.e.* the exponents in the different groups may be regarded as the halves of the numbers which are the products of (i) two uneven numbers, (ii) an uneven and an even number, (iii) an even and an uneven number, (iv) two even numbers.

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§ 40. The apparent simplicity of the form of the series in § 38 is due to the arbitrary convention made with respect to the meaning of q^n. Otherwise these series are not expressible as a double summation of a single term.

If we express the series entirely by means of exponentials, the formulæ may be written;

$$\rho\,\mathrm{ns}\,\rho x = \mathrm{cec}\,x - 2i\Sigma_1^\infty \Sigma_1^\infty \{e^{-s_1(\frac{1}{2}r_2 - ix)} - e^{-s_1(\frac{1}{2}r_2 + ix)}\},$$

$$\rho\,\mathrm{ds}\,\rho x = \mathrm{cec}\,x - 2i\Sigma_1^\infty \Sigma_1^\infty (-)^r \{e^{-s_1(\frac{1}{2}r_2 - ix)} - e^{-s_1(\frac{1}{2}r_2 + ix)}\},$$

&c. &c.

Series proceeding by powers of q' and e^x, § 41.

§ 41. By expanding the q'-coefficients in § 22 (p. 18) in ascending powers of q', and arranging the formulæ in groups according to the form of the series, we find

$$k'\rho'\,\mathrm{sc}\,\rho' x = 4\Sigma_1^\infty \Sigma_1^\infty q'^{\frac{1}{2}r_1 s_1} \sinh s_1 x,$$

$$k'\rho'\,\mathrm{nc}\,\rho' x = -4\Sigma_1^\infty \Sigma_1^\infty (-)^r q'^{\frac{1}{2}r_1 s_1} \cosh s_1 x,$$

$$k'\rho'\,\mathrm{nd}\,\rho' x = -4\Sigma_1^\infty \Sigma_1^\infty (-)^s q'^{\frac{1}{2}r_1 s_1} \cosh s_1 x,$$

$$kk'\rho'\,\mathrm{sd}\,\rho' x = 4\Sigma_1^\infty \Sigma_1^\infty (-)^{r+s} q'^{\frac{1}{2}r_1 s_1} \sinh s_1 x;$$

$$\rho'\,\mathrm{zc}\,\rho' x = -\frac{2x}{\mu'} - 4\Sigma_1^\infty \Sigma_1^\infty q'^{\frac{1}{2}r_1 s_2} \sinh s_2 x,$$

$$\rho'\,\mathrm{dc}\,\rho' x = 1 - 4\Sigma_1^\infty \Sigma_1^\infty (-)^r q'^{\frac{1}{2}r_1 s_2} \cosh s_2 x,$$

$$\rho'\,\mathrm{zd}\,\rho' x = -\frac{2x}{\mu'} - 4\Sigma_1^\infty \Sigma_1^\infty (-)^s q'^{\frac{1}{2}r_1 s_2} \sinh s_2 x,$$

$$k\rho'\,\mathrm{cd}\,\rho' x = 1 - 4\Sigma_1^\infty \Sigma_1^\infty (-)^{r+s} q'^{\frac{1}{2}r_1 s_2} \cosh s_2 x;$$

$$\rho'\,\mathrm{cs}\,\rho' x = \mathrm{cech}\,x - 4\Sigma_1^\infty \Sigma_1^\infty q'^{\frac{1}{2}r_2 s_1} \sinh s_1 x,$$

$$\rho'\,\mathrm{ds}\,\rho' x = \mathrm{cech}\,x - 4\Sigma_1^\infty \Sigma_1^\infty (-)^r q'^{\frac{1}{2}r_2 s_1} \sinh s_1 x,$$

$$\rho'\,\mathrm{dn}\,\rho' x = \mathrm{sech}\,x - 4\Sigma_1^\infty \Sigma_1^\infty (-)^s q'^{\frac{1}{2}r_2 s_1} \cosh s_1 x,$$

$$k\rho'\,\mathrm{cn}\,\rho' x = \mathrm{sech}\,x - 4\Sigma_1^\infty \Sigma_1^\infty (-)^{r+s} q'^{\frac{1}{2}r_2 s_1} \cosh s_1 x;$$

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$$\rho' \operatorname{zs}\rho' x = -\frac{2x}{\mu'} + \coth x - 4\Sigma_1^\infty \Sigma_1^\infty q'^{\frac{1}{2}r_2 s_2} \sinh s_2 x,$$

$$\rho' \operatorname{ns}\rho' x = \qquad \coth x - 4\Sigma_1^\infty \Sigma_1^\infty (-)^r q'^{\frac{1}{2}r_2 s_2} \sinh s_2 x,$$

$$\rho' \operatorname{zn}\rho' x = -\frac{2x}{\mu'} + \tanh x - 4\Sigma_1^\infty \Sigma_1^\infty (-)^s q'^{\frac{1}{2}r_2 s_2} \sinh s_2 x,$$

$$k\rho' \operatorname{sn}\rho' x = \qquad \tanh x - 4\Sigma_1^\infty \Sigma_1^\infty (-)^{r+s} q'^{\frac{1}{2}r_2 s_2} \sinh s_2 x.$$

The notation is the same as in § 39.

These formulæ may be expressed in a form corresponding to that in § 38 by adopting a similar convention with respect to the meaning of q'^{-n}.

Series of the forms $\Sigma_{-\infty}^{\infty} \Sigma_{-\infty}^{\infty} \frac{e^{\lambda i x}}{r\mu + si\pi}$, &c., §§ 42–47.

§ 42. If in the formulæ of § 20 (pp. 16, 17) we substitute for the hyperbolic functions their series-values given by the formulæ:

$$\coth r\mu = \Sigma_{-\infty}^{\infty} \frac{1}{r\mu + \frac{1}{2}s_2 i\pi}, \qquad \tanh r\mu = \Sigma_{-\infty}^{\infty} \frac{1}{r\mu + \frac{1}{2}s_1 i\pi},$$

$$\operatorname{cech} r\mu = \Sigma_{-\infty}^{\infty} (-)^s \frac{1}{r\mu + \frac{1}{2}s_2 i\pi}, \quad \operatorname{sech} r\mu = -i\Sigma_{-\infty}^{\infty} (-)^s \frac{1}{r\mu + \frac{1}{2}s_1 i\pi},$$

we obtain a system of double summations which may be written as follows:

$$\rho \operatorname{ns}\rho x = \frac{\cos(v_1 + 1)x}{\sin x} + 2\Sigma_{-v_1}^{v_1} \Sigma_{-\infty}^{\infty} \frac{z^{s_1}}{r_2\pi + s_1 i\mu},$$

$$\rho \operatorname{ds}\rho x = \frac{\cos(v_1 + 1)x}{\sin x} + 2\Sigma_{-v_1}^{v_1} \Sigma_{-\infty}^{\infty} \frac{z^{s_1}}{r_1\pi + s_1 i\mu},$$

$$\rho \operatorname{cs}\rho x = \frac{\cos(v_2 + 1)x}{\sin x} + 2\Sigma_{-v_2}^{v_2} \Sigma_{-\infty}^{\infty} \frac{z^{s_2}}{r_1\pi + s_2 i\mu},$$

$$\rho \operatorname{zs}\rho x = \frac{\cos(v_2 + 1)x}{\sin x} + 2\Sigma_{-v_2}^{v_2} \Sigma_{-\infty}^{\infty} \frac{z^{s_2}}{r_2\pi + s_2 i\mu};$$

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$$\rho\,\mathrm{dc}\,\rho x=(-)^{v}\ \frac{\cos(v_1+1)x}{\cos x}-2i\Sigma_{-v_1}^{v_1}\Sigma_{-\infty}^{\infty}(-)^{s}\frac{z^{s_1}}{r_2\pi+s_1 i\mu},$$

$$k'\rho\,\mathrm{nc}\,\rho x=(-)^{v}\ \frac{\cos(v_1+1)x}{\cos x}-2i\Sigma_{-v_1}^{v_1}\Sigma_{-\infty}^{\infty}(-)^{s}\frac{z^{s_1}}{r_1\pi+s_1 i\mu},$$

$$k'\rho\,\mathrm{sc}\,\rho x=(-)^{v}\ \frac{\sin(v_2+1)x}{\cos x}-2\,\Sigma_{-v_2}^{v_2}\Sigma_{-\infty}^{\infty}(-)^{s}\frac{z^{s_2}}{r_1\pi+s_2 i\mu},$$

$$\rho\,\mathrm{zc}\,\rho x=(-)^{v+1}\frac{\sin(v_2+1)x}{\cos x}+2\,\Sigma_{-v_2}^{v_2}\Sigma_{-\infty}^{\infty}(-)^{s}\frac{z^{s_2}}{r_2\pi+s_2 i\mu};$$

$$k\rho\,\mathrm{cd}\,x=-2i\Sigma_{-\infty}^{\infty}\Sigma_{-\infty}^{\infty}(-)^{r+s}\frac{z^{s_1}}{r_2\pi+s_1 i\mu},$$

$$kk'\rho\,\mathrm{sd}\,x=-2i\Sigma_{-\infty}^{\infty}\Sigma_{-\infty}^{\infty}(-)^{r+s}\frac{z^{s_1}}{r_1\pi+s_1 i\mu},$$

$$k'\rho\,\mathrm{nd}\,x=\ 2\,\Sigma_{-\infty}^{\infty}\Sigma_{-\infty}^{\infty}(-)^{r+s}\frac{z^{s_2}}{r_1\pi+s_2 i\mu},$$

$$\rho\,\mathrm{zd}\,x=-2\,\Sigma_{-\infty}^{\infty}\Sigma_{-\infty}^{\infty}(-)^{r+s}\frac{z^{s_2}}{r_2\pi+s_2 i\mu},$$

$$k\rho\,\mathrm{sn}\,x=-2\Sigma_{-\infty}^{\infty}\Sigma_{-\infty}^{\infty}(-)^{r}\frac{z^{s_1}}{r_2\pi+s_1 i\mu},$$

$$k\rho\,\mathrm{cn}\,x=\ 2\Sigma_{-\infty}^{\infty}\Sigma_{-\infty}^{\infty}(-)^{r}\frac{z^{s_1}}{r_1\pi+s_1 i\mu},$$

$$\rho\,\mathrm{dn}\,x=-2\Sigma_{-\infty}^{\infty}\Sigma_{-\infty}^{\infty}(-)^{r}\frac{z^{s_2}}{r_1\pi+s_2 i\mu},$$

$$\rho\,\mathrm{zn}\,x=-2\Sigma_{-\infty}^{\infty}\Sigma_{-\infty}^{\infty}(-)^{r}\frac{z^{s_2}}{r_2\pi+s_2 i\mu}.$$

The notation is exactly the same as in §§ 38 and 39. The infinite limits $\pm v_1$ and $\pm v_2$ refer to s_1 and s_2.*

* Thus v_1 and v_2 have the same meaning as w and $2v$ in § 20, viz. v_1 and v_2 denote an infinitely large uneven integer and an infinitely large even integer respectively.

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In the above formulæ the infinite limits of r_1 and r_2 are supposed to be infinitely greater than those of s_1 and s_2 (*i.e.* than v_1 and v_2). This is a consequence of the mode of their derivation from the formulæ in § 20.

If the limits of s_1 and s_2 are supposed infinitely greater than those of r_1 and r_2, the terms $\frac{\cos(v_1+1)x}{\sin x}$, &c. which occur in the first two groups disappear. This will be seen to be the case in § 45.

§ 43. By combining equal positive and negative values of r_1, r_2, s_1, s_2 we at once transform the preceding system into the following, in which, as before, the r-limits are supposed to be infinitely greater than the s-limits.

$$\rho\,\mathrm{ns}\rho x = \frac{\cos(v_1+1)x}{\sin x} + \frac{4}{\mu}\Sigma_1^{v_1}\frac{\sin s_1 x}{s_1} + 8\mu\Sigma_1^{v_1}\Sigma_2^{\infty}\frac{s_1\sin s_1 x}{r_2{}^2\pi^2 + s_1{}^2\mu^2},$$

$$\rho\,\mathrm{ds}\rho x = \frac{\cos(v_1+1)x}{\sin x} + 8\mu\Sigma_1^{v_1}\Sigma_1^{\infty}\frac{s_1\sin s_1 x}{r_1{}^2\pi^2 + s_1{}^2\mu^2},$$

$$\rho\,\mathrm{cs}\rho x = \frac{\cos(v_2+1)x}{\sin x} + 8\mu\Sigma_2^{v_2}\Sigma_1^{\infty}\frac{s_2\sin s_2 x}{r_1{}^2\pi^2 + s_2{}^2\mu^2},$$

$$\rho\,\mathrm{zs}\rho x = \frac{\cos(v_2+1)x}{\sin x} + \frac{4}{\mu}\Sigma_2^{v_2}\frac{\sin s_2 x}{s_2} + 8\mu\Sigma_2^{v_2}\Sigma_2^{\infty}\frac{s_2\sin s_2 x}{r_2{}^2\pi^2 + s_2{}^2\mu^2};$$

$$\rho\,\mathrm{dc}\rho x = (-)^v\frac{\cos(v_1+1)x}{\cos x} - \frac{4}{\mu}\Sigma_1^{v_1}(-)^s\frac{\sin s_1 x}{s_1} - 8\mu\Sigma_1^{v_1}\Sigma_2^{\infty}(-)^s\frac{s_1\cos s_1 x}{r_2{}^2\pi^2 + s_1{}^2\mu^2},$$

$$k'\rho\,\mathrm{nc}\rho x = (-)^v\frac{\cos(v_1+1)x}{\cos x} - 8\mu\Sigma_1^{v_1}\Sigma_1^{\infty}(-)^s\frac{s_1\cos s_1 x}{r_1{}^2\pi^2 + s_1{}^2\mu^2},$$

$$k'\rho\,\mathrm{sc}\rho x = (-)^v\frac{\sin(v_2+1)x}{\cos x} - 8\mu\Sigma_2^{v_2}\Sigma_1^{\infty}(-)^s\frac{s_2\sin s_2 x}{r_1{}^2\pi^2 + s_2{}^2\mu^2},$$

$$\rho\,\mathrm{zc}\rho x = (-)^{v+1}\frac{\sin(v_2+1)x}{\cos x} + \frac{4}{\mu}\Sigma_2^{v_2}(-)^s\frac{\sin s_2 x}{s_2} + 8\mu\Sigma_2^{v_2}\Sigma_2^{\infty}(-)^s\frac{s_2\sin s_2 x}{r_2{}^2\pi^2 + s_2{}^2\mu^2};$$

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$$k\rho\ \mathrm{cd}\,\rho x = -\frac{4}{\mu}\Sigma_1^{v_1}(-)^s\frac{\cos s_1 x}{s_1} - 8\mu\Sigma_1^\infty\Sigma_2^\infty(-)^{r+s}\frac{s_1\cos s_1 x}{r_2^2\pi^2+s_1^2\mu^2},$$

$$kk'\rho\ \mathrm{sd}\,\rho x = 8\pi\Sigma_1^\infty\Sigma_1^\infty(-)^{r+s}\frac{r_1\sin s_1 x}{r_1^2\pi^2+s_1^2\mu^2},$$

$$k'\rho\ \mathrm{nd}\,\rho x = 1 - 8\pi\Sigma_2^\infty\Sigma_1^\infty(-)^{r+s}\frac{r_1\cos s_2 x}{r_1^2\pi^2+s_2^2\mu^2},$$

$$\rho\ \mathrm{zd}\,\rho x = \frac{4}{\mu}\Sigma_2^{v_2}(-)^s\frac{\sin s_2 x}{s_2} + 8\mu\Sigma_2^\infty\Sigma_2^\infty(-)^{r+s}\frac{s_2\sin s_2 x}{r_2^2\pi^2+s_2^2\mu^2},$$

$$k\rho\ \mathrm{sn}\,\rho x = \frac{4}{\mu}\Sigma_1^{v_1}\frac{\sin s_1 x}{s_1} + 8\mu\Sigma_1^\infty\Sigma_2^\infty(-)^r\frac{s_1\sin s_1 x}{r_2^2\pi^2+s_1^2\mu^2},$$

$$k\rho\ \mathrm{cn}\,\rho x = -8\pi\Sigma_1^\infty\Sigma_1^\infty(-)^r\frac{r_1\cos s_1 x}{r_1^2\pi^2+s_1^2\mu^2},$$

$$\rho\ \mathrm{dn}\,\rho x = 1 - 8\pi\Sigma_2^\infty\Sigma_1^\infty(-)^r\frac{r_1\cos s_2 x}{r_1^2\pi^2+s_2^2\mu^2},$$

$$\rho\ \mathrm{zn}\,\rho x = \frac{4}{\mu}\Sigma_2^{v_2}\frac{\sin s_2 x}{s_2} + 8\mu\Sigma_2^\infty\Sigma_2^\infty(-)^r\frac{s_2\sin s_2 x}{r_2^2\pi^2+s_2^2\mu^2}.$$

§ 44. We may effect the summation of the preceding double series with respect to s_1 or s_2 by means of the formulæ

$$\Sigma_1^\infty\frac{s_1\sin s_1 x}{a^2+s_1^2} = \tfrac{1}{4}\pi\frac{\cosh a(\tfrac{1}{2}\pi - x)}{\cosh\tfrac{1}{2}a\pi},$$

$$\Sigma_2^\infty\frac{s_2\sin s_2 x}{a^2+s_2^2} = \tfrac{1}{4}\pi\frac{\sinh a(\tfrac{1}{2}\pi - x)}{\sinh\tfrac{1}{2}a\pi},$$

$$\Sigma_1^\infty\frac{a\cos s_1 x}{a^2+s_1^2} = \tfrac{1}{4}\pi\frac{\sinh a(\tfrac{1}{2}\pi - x)}{\cosh\tfrac{1}{2}a\pi},$$

$$\Sigma_2^\infty\frac{a\cos s_2 x}{a^2+s_2^2} = \tfrac{1}{4}\pi\frac{\cosh a(\tfrac{1}{2}\pi - x)}{\sinh\tfrac{1}{2}a\pi} - \frac{1}{2a},$$

$$\Sigma_1^\infty\frac{\sin s_1 x}{s_1} = \tfrac{1}{4}\pi,$$

$$\Sigma_2^\infty\frac{\sin s_2 x}{s_2} = \tfrac{1}{4}\pi - \tfrac{1}{2}x,$$

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which are true when x lies between 0 and π, and the formulæ

$$\Sigma_1^\infty (-)^s \frac{s_1 \cos s_1 x}{a^2 + s_1^2} = -\tfrac{1}{2}\pi \frac{\cosh ax}{\cosh \frac{1}{2}a\pi},$$

$$\Sigma_2^\infty (-)^s \frac{s_2 \sin s_2 x}{a^2 + s_2^2} = -\tfrac{1}{2}\pi \frac{\sinh ax}{\sinh \frac{1}{2}a\pi},$$

$$\Sigma_1^\infty (-)^s \frac{a \sin s_1 x}{a^2 + s_1^2} = -\tfrac{1}{2}\pi \frac{\sinh ax}{\cosh \frac{1}{2}a\pi},$$

$$\Sigma_2^\infty (-)^s \frac{a \cos s_2 x}{a^2 + s_2^2} = \tfrac{1}{2}\pi \frac{\cosh ax}{\cosh \frac{1}{2}a\pi} - \frac{1}{2a},$$

$$\Sigma_1^\infty (-)^s \frac{\cos s_1 x}{s_1} = -\tfrac{1}{2}\pi,$$

$$\Sigma_2^\infty (-)^s \frac{\sin s_2 x}{s_2} = -\tfrac{1}{2}x,$$

which are true when x lies between $\pm \frac{1}{2}\pi$.

The last two formulæ in each of these groups are deducible from the first two by putting $a = 0$.

§ 45. If we ignore the terms $\frac{\cos(v_1+1)x}{\sin x}$, &c. and sum the series with respect to s_1 and s_2, we find

$$\rho\, \mathrm{ns}\rho x = \frac{\pi}{\mu}\left\{1 + 2\Sigma_2^\infty \frac{\cosh r_2(\frac{1}{2}\mu' - X)}{\cosh \frac{1}{2}r_2\mu'}\right\},$$

$$\rho\, \mathrm{ds}\rho x = 2\frac{\pi}{\mu}\Sigma_1^\infty \frac{\cosh r_1(\frac{1}{2}\mu' - X)}{\cosh \frac{1}{2}r_1\mu'},$$

$$\rho\, \mathrm{cs}\rho x = 2\frac{\pi}{\mu}\Sigma_1^\infty \frac{\sinh r_1(\frac{1}{2}\mu' - X)}{\sinh \frac{1}{2}r_1\mu'},$$

$$\rho\, \mathrm{zs}\rho x = \frac{\pi}{\mu}\left\{1 - \frac{2x}{\pi} + 2\Sigma_2^\infty \frac{\sinh r_2(\frac{1}{2}\mu' - X)}{\sinh \frac{1}{2}r_2\mu'}\right\},$$

&c. &c.,

where $\mu' = \frac{\pi^2}{\mu}$ and $X = \frac{\pi x}{\mu}$.

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In the first group, which has alone been written, x is restricted to be within the limits $\pm\frac{1}{2}\mu$. The limits are the same in the fourth group. In the second and third groups they are 0 and π.

This system of formulæ is the same as that contained in § 24 (p. 20).

By effecting the summation with respect to s_1 or s_2 we assume that the limits of s_1 and s_2 are absolute infinities, *i.e.* that they are infinitely greater than the limits of r_1 and r_2. Since we thus arrive at the system of formulæ in § 24 by ignoring the term $\dfrac{\cos(v_1+1)x}{\sin x}$, &c. we may infer that when the s-limits are infinitely greater than the r-limits, these terms disappear from the formulæ given in §§ 42 and 43, which apply to the case where the r-limits are infinitely greater than the s-limits.

§ 46. By putting ix for x and changing the modulus from k to k', we find from § 42,

$$\rho' \operatorname{ns}\rho' x = \tfrac{1}{2}\frac{e^{(v_2+1)x}}{\sinh x} - 2\Sigma_{-v_2}^{v_2}\Sigma_{-\infty}^{\infty}\frac{e^{s_2 x}}{s_2\mu' + r_1 i\pi},$$

$$\rho' \operatorname{ds}\rho' x = \tfrac{1}{2}\frac{e^{(v_1+1)x}}{\sinh x} - 2\Sigma_{-v_1}^{v_1}\Sigma_{-\infty}^{\infty}\frac{e^{s_1 x}}{s_1\mu' + r_1 i\pi},$$

$$\rho' \operatorname{cs}\rho' x = \tfrac{1}{2}\frac{e^{(v_1+1)x}}{\sinh x} - 2\Sigma_{-v_1}^{v_1}\Sigma_{-\infty}^{\infty}\frac{e^{s_1 x}}{s_1\mu' + r_2 i\pi},$$

$$\rho' \operatorname{zs}\rho' x = -\frac{2x}{\mu'} + \tfrac{1}{2}\frac{e^{(v_2+1)x}}{\sinh x} - 2\Sigma_{-v_2}^{v_2}\Sigma_{-\infty}^{\infty}\frac{e^{s_2 x}}{s_2\mu' + r_2 i\pi},$$

&c. &c.

The r-limits are supposed to be infinitely greater than the s-limits, v_1 and v_2. If the s-limits are infinitely greater than the r-limits, the terms $\frac{1}{2}\dfrac{e^{(v_2+1)x}}{\sinh x}$, &c. disappear.

These formulæ may also be deduced directly from § 23 (p. 19).

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§ 47. It may be remarked that by putting for μ its value in terms of K and K' we have, from § 42,

$$k \operatorname{sn} x = -\Sigma_{-\infty}^{\infty}\Sigma_{-\infty}^{\infty}(-)^r \frac{z^{s_1}}{r_2 K + s_1 i K'},$$

$$k \operatorname{cn} x = \Sigma_{-\infty}^{\infty}\Sigma_{-\infty}^{\infty}(-)^r \frac{z^{s_1}}{r_1 K + s_1 i K'},$$

$$\operatorname{dn} x = -\Sigma_{-\infty}^{\infty}\Sigma_{-\infty}^{\infty}(-)^r \frac{z^{s_2}}{r_1 K + s_2 i K'},$$

$$\operatorname{zn} x = -\Sigma_{-\infty}^{\infty}\Sigma_{-\infty}^{\infty}(-)^r \frac{z^{s_2}}{r_2 K + s_2 i K'},$$

with three similar groups relating to the other functions. If the r-limits are infinitely greater than the s-limits the additional terms $\frac{1}{\rho}\frac{\cos(v,+1)x}{\sin x}$, &c. make their appearance in two of the groups.

Series of the forms $\Sigma_{-\infty}^{\infty}\Sigma_{-\infty}^{\infty}\frac{1}{x + r\pi + si\mu}$, &c., §§ 48, 49.

§ 48. By substituting for the circular functions in § 15, or or for the hyperbolic functions in § 16, their values as series of the form $\Sigma_{-\infty}^{\infty}\phi(x + na)$, we obtain a system of double summations for the elliptic functions. These expressions may be regarded as analogues of the trigonometrical formulæ:

$$\cot x = \Sigma_{-\infty}^{\infty} \frac{1}{x + \frac{1}{2}r_2\pi},$$

$$\operatorname{cec} x = \Sigma_{-\infty}^{\infty}(-)^r \frac{1}{x + \frac{1}{2}r_2\pi},$$

$$\tan x = -\Sigma_{-\infty}^{\infty} \frac{1}{x + \frac{1}{2}r_1\pi},$$

$$\sec x = -\Sigma_{-\infty}^{\infty}(-)^r \frac{1}{x + \frac{1}{2}r_1\pi},$$

and of the corresponding formulæ for $\coth x$, &c.

§ 49. The complete system of sixteen formulæ is given on the next page. The notation is exactly the same as in the last six sections.

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$$\rho\,\mathrm{zd}\,\rho x = \Sigma_{-\infty}^{\infty}\Sigma_{-\infty}^{\infty}\frac{1}{x+\frac{1}{2}r_1\pi+\frac{1}{2}s_1 i\mu},$$

$$k\rho\,\mathrm{cd}\,\rho x = -\Sigma_{-\infty}^{\infty}\Sigma_{-\infty}^{\infty}(-)^r\frac{1}{x+\frac{1}{2}r_1\pi+\frac{1}{2}s_1 i\mu},$$

$$k'\rho\,\mathrm{nd}\,\rho x = 1 - i\Sigma_{-\infty}^{\infty}\Sigma_{-\infty}^{\infty}(-)^s\frac{1}{x+\frac{1}{2}r_1\pi+\frac{1}{2}s_1 i\mu},$$

$$kk'\rho\,\mathrm{sd}\,\rho x = -i\Sigma_{-\infty}^{\infty}\Sigma_{-\infty}^{\infty}(-)^{r+s}\frac{1}{x+\frac{1}{2}r_1\pi+\frac{1}{2}s_1 i\mu};$$

$$\rho\,\mathrm{zc}\,\rho x = \Sigma_{-\infty}^{\infty}\Sigma_{-\infty}^{\infty}\frac{1}{x+\frac{1}{2}r_1\pi+\frac{1}{2}s_2 i\mu},$$

$$\rho\,\mathrm{dc}\,\rho x = -\Sigma_{-\infty}^{\infty}\Sigma_{-\infty}^{\infty}(-)^r\frac{1}{x+\frac{1}{2}r_1\pi+\frac{1}{2}s_2 i\mu},$$

$$k'\rho\,\mathrm{sc}\,\rho x = -\Sigma_{-\infty}^{\infty}\Sigma_{-\infty}^{\infty}(-)^s\frac{1}{x+\frac{1}{2}r_1\pi+\frac{1}{2}s_2 i\mu},$$

$$k'\rho\,\mathrm{nc}\,\rho x = -\Sigma_{-\infty}^{\infty}\Sigma_{-\infty}^{\infty}(-)^{r+s}\frac{1}{x+\frac{1}{2}r_1\pi+\frac{1}{2}s_2 i\mu};$$

$$\rho\,\mathrm{zn}\,\rho x = \Sigma_{-\infty}^{\infty}\Sigma_{-\infty}^{\infty}\frac{1}{x+\frac{1}{2}r_2\pi+\frac{1}{2}s_1 i\mu},$$

$$k\rho\,\mathrm{sn}\,\rho x = \Sigma_{-\infty}^{\infty}\Sigma_{-\infty}^{\infty}(-)^r\frac{1}{x+\frac{1}{2}r_2\pi+\frac{1}{2}s_1 i\mu},$$

$$\rho\,\mathrm{dn}\,\rho x = 1 - i\Sigma_{-\infty}^{\infty}\Sigma_{-\infty}^{\infty}(-)^s\frac{1}{x+\frac{1}{2}r_2\pi+\frac{1}{2}s_1 i\mu},$$

$$k\rho\,\mathrm{cn}\,\rho x = -i\Sigma_{-\infty}^{\infty}\Sigma_{-\infty}^{\infty}(-)^{r+s}\frac{1}{x+\frac{1}{2}r_2\pi+\frac{1}{2}s_1 i\mu};$$

$$\rho\,\mathrm{zs}\,\rho x = \Sigma_{-\infty}^{\infty}\Sigma_{-\infty}^{\infty}\frac{1}{x+\frac{1}{2}r_2\pi+\frac{1}{2}s_2 i\mu},$$

$$\rho\,\mathrm{ns}\,\rho x = \Sigma_{-\infty}^{\infty}\Sigma_{-\infty}^{\infty}(-)^r\frac{1}{x+\frac{1}{2}r_2\pi+\frac{1}{2}s_2 i\mu},$$

$$\rho\,\mathrm{cs}\,\rho x = \Sigma_{-\infty}^{\infty}\Sigma_{-\infty}^{\infty}(-)^s\frac{1}{x+\frac{1}{2}r_2\pi+\frac{1}{2}s_2 i\mu},$$

$$\rho\,\mathrm{ds}\,\rho x = \Sigma_{-\infty}^{\infty}\Sigma_{-\infty}^{\infty}(-)^{r+s}\frac{1}{x+\frac{1}{2}r_2\pi+\frac{1}{2}s_2 i\mu}.$$

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In each equation the corresponding positive and negative infinite limits are supposed to be exactly equal. In the series for $k'\rho\,\mathrm{nd}\,\rho x$ and $\rho\,\mathrm{dn}\,\rho x$ the infinite limits of s_1 are supposed to be of the form $\pm(4n+3)$.

In the above formulæ the r-limits are supposed to be infinitely larger than the s-limits. If the s-limits are infinitely larger than the r-limits the term unity is to be transferred from the series for $k'\rho\,\mathrm{nd}\,\rho x$ and $\rho\,\mathrm{dn}\,\rho x$ to those for $k\rho\,\mathrm{cd}\,\rho x$ and $\rho\,\mathrm{dc}\,\rho x$; the restriction of the form of the s_1-limits of $k'\rho\,\mathrm{nd}\,\rho x$ and $\rho\,\mathrm{dn}\,\rho x$ being also transferred to the r_1-limits of $k\rho\,\mathrm{cd}\,\rho x$ and $\rho\,\mathrm{dc}\,\rho x$. In the case of each of the Zeta functions the term $-\frac{2x}{\mu}$ is to be added.

Systems of identical equations, §§ 50, 51.

§ 50. Each of the groups of series considered in the last section gives rise to interesting groups of identical equations.

Taking the last group, and using D to denote $\frac{d}{dx}$, since

$$D\rho\,\mathrm{ns}\,\rho x = -\rho\,\mathrm{ds}\,\rho x.\rho\,\mathrm{cs}\,\rho x,$$

$$D\rho\,\mathrm{ds}\,\rho x = -\rho\,\mathrm{cs}\,\rho x.\rho\,\mathrm{ns}\,\rho x,$$

$$D\rho\,\mathrm{cs}\,\rho x = -\rho\,\mathrm{ns}\,\rho x.\rho\,\mathrm{ds}\,\rho x,$$

$$D^2\rho\,\mathrm{zs}\,\rho x = 2\rho\,\mathrm{ns}\,\rho x.\rho\,\mathrm{ds}\,\rho x.\rho\,\mathrm{cs}\,\rho x,$$

we find, by substituting for the functions their values, the following curious identities, in which, for brevity, (x, r, s) denotes $x + r\pi + si\mu$ and Σ represents $\Sigma_{-\infty}^{\infty}\Sigma_{-\infty}^{\infty}$:

$$\Sigma(-)^{r+s}\frac{1}{(x,r,s)^2} = \Sigma(-)^r\frac{1}{(x,r,s)}\,\Sigma(-)^s\frac{1}{(x,r,s)},$$

$$\Sigma(-)^r\frac{1}{(x,r,s)^2} = \Sigma(-)^s\frac{1}{(x,r,s)}\,\Sigma(-)^{r+s}\frac{1}{(x,r,s)},$$

$$\Sigma(-)^s\frac{1}{(x,r,s)^2} = \Sigma(-)^r\frac{1}{(x,r,s)}\,\Sigma(-)^{r+s}\frac{1}{(x,r,s)},$$

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and

$$\Sigma \frac{1}{(x, r, s)^3} = \Sigma(-)^r \frac{1}{(x, r, s)} \Sigma(-)^r \frac{1}{(x, r, s)^2}$$

$$= \Sigma(-)^s \frac{1}{(x, r, s)} \Sigma(-)^s \frac{1}{(x, r, s)^2} = \Sigma(-)^{n+r} \frac{1}{(x, r, s)} \Sigma(-)^{n+r} \frac{1}{(x, r, s)^2}$$

$$= \Sigma(-)^r \frac{1}{(x, r, s)} \Sigma(-)^s \frac{1}{(x, r, s)} \Sigma(-)^{r+s} \frac{1}{(x, r, s)},$$

The letters r and s are each to have all integral values from $-\infty$ to ∞.

The values of the series are independent of the ratio of the r-limits and the s-limits.

These relations correspond to the trigonometrical formula

$$\Sigma_{-\infty}^{\infty}(-)^n \frac{1}{(x+n\pi)^2} = \Sigma_{-\infty}^{\infty} \frac{1}{x+n\pi} \Sigma_{-\infty}^{\infty}(-)^n \frac{1}{x+n\pi},$$

$$\Sigma_{-\infty}^{\infty} \frac{1}{(x+n\pi)^2} = \left\{\Sigma_{-\infty}^{\infty}(-)^n \frac{1}{x+n\pi}\right\}^2.$$

§ 51. It may be noticed that the following remarkable identical relations, which are deducible from § 15 in the same manner, may also be regarded as analogues of the above trigonometrical formulæ:

$$\Sigma_{-\infty}^{\infty} \operatorname{cec}(x+ni\mu) \cot(x+ni\mu)$$
$$= \Sigma_{-\infty}^{\infty}(-)^n \operatorname{cec}(x+ni\mu) \Sigma_{-\infty}^{\infty}(-)^n \cot(x+ni\mu),$$

$$\Sigma_{-\infty}^{\infty}(-)^n \operatorname{cec}(x+ni\mu) \cot(x+ni\mu)$$
$$= \Sigma_{-\infty}^{\infty} \operatorname{cec}(x+ni\mu) \Sigma_{-\infty}^{\infty}(-)^n \cot(x+ni\mu),$$

$$\Sigma_{-\infty}^{\infty}(-)^n \operatorname{cec}^2(x+ni\mu)$$
$$= \Sigma_{-\infty}^{\infty} \operatorname{cec}(x+ni\mu) \Sigma_{-\infty}^{\infty}(-)^n \operatorname{cec}(x+ni\mu),$$

and

$$\Sigma_{-\infty}^{\infty} \operatorname{cec}^2(x+ni\mu) \cot(x+ni\mu)$$
$$= \Sigma_{-\infty}^{\infty}(-)^n \operatorname{cec}^2(x+ni\mu) \Sigma_{-\infty}^{\infty}(-)^n \cot(x+ni\mu)$$
$$= \Sigma_{-\infty}^{\infty}(-)^n \operatorname{cec}(x+ni\mu) \Sigma_{-\infty}^{\infty}(-)^n \operatorname{cec}(x+ni\mu) \cot(x+ni\mu)$$
$$= \Sigma_{-\infty}^{\infty} \operatorname{cec}(x+ni\mu) \Sigma_{-\infty}^{\infty} \operatorname{cec}(x+ni\mu) \cot(x+ni\mu)$$
$$= \Sigma_{-\infty}^{\infty} \operatorname{cec}(x+ni\mu) \Sigma_{-\infty}^{\infty}(-)^n \operatorname{cec}(x+ni\mu) \Sigma_{-\infty}^{\infty}(-)^n \cot(x+ni\mu).$$

In these formulæ x and μ may be any real quantities; or any complex quantities which do not render the series divergent.

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Thus the three double series

$$\Sigma_{-\infty}^{\infty}\Sigma_{-\infty}^{\infty}(-)^{r}\frac{1}{x+r\pi+si\mu},$$

$$\Sigma_{-\infty}^{\infty}\Sigma_{-\infty}^{\infty}(-)^{s}\frac{1}{x+r\pi+si\mu},$$

$$\Sigma_{-\infty}^{\infty}\Sigma_{-\infty}^{\infty}(-)^{r+s}\frac{1}{x+r\pi+si\mu},$$

and also the three single series

$\Sigma_{-\infty}^{\infty}\operatorname{cec}(x+ni\mu)$, $\Sigma_{-\infty}^{\infty}(-)^{n}\operatorname{cec}(x+ni\mu)$, $\Sigma_{-\infty}^{\infty}(-)^{n}\cot(x+ni\mu)$

possess the singular properties typified by the equations

$$\Sigma a\,\Sigma b \quad = \Sigma ab,$$

$$\Sigma a\,\Sigma b\,\Sigma c = \Sigma abc.$$

Expansions in powers of q and x, § 52.

§ 52. By expanding in ascending powers of x the functions $\Delta(\sin mx)$, $E(\cos mx)$, &c., which occur in the formulæ in § 9, we obtain the following system of doubly infinite series.

$$k\rho\,\operatorname{sn}\rho x = 4\Sigma_{1}^{\infty}\Sigma_{1}^{\infty}(-)^{t-1}\Delta_{2t-1}(m)q^{\frac{1}{2}m}\frac{x^{2t-1}}{(2t-1)!},$$

$$kk'\rho\,\operatorname{sd}\rho x = 4\Sigma_{1}^{\infty}\Sigma_{1}^{\infty}(-)^{t-1+M}\Delta_{2t-1}(m)q^{\frac{1}{2}m}\frac{x^{2t-1}}{(2t-1)!},$$

$$k\rho\,\operatorname{cd}\rho x = 4\Sigma_{1}^{\infty}\Sigma_{1}^{\infty}(-)^{t-1}E_{2t-2}(m)q^{\frac{1}{2}m}\frac{x^{2t-2}}{(2t-2)!},$$

$$k\rho\,\operatorname{cn}\rho x = 4\Sigma_{1}^{\infty}\Sigma_{1}^{\infty}(-)^{t-1+M}E_{2t-2}(m)q^{\frac{1}{2}m}\frac{x^{2t-2}}{(2t-2)!};$$

$$\rho\,\operatorname{zn}\rho x = 4\Sigma_{1}^{\infty}\Sigma_{1}^{\infty}(-)^{t-1}2^{2t-1}\Delta'_{2t-1}(n)q^{n}\frac{x^{2t-1}}{(2t-1)!},$$

$$\rho\,\operatorname{zd}\rho x = 4\Sigma_{1}^{\infty}\Sigma_{1}^{\infty}(-)^{t-1+n}2^{2t-1}\Delta'_{2t-1}(n)q^{n}\frac{x^{2t-1}}{(2t-1)!},$$

$$\rho\,\operatorname{dn}\rho x = 1+4\Sigma_{1}^{\infty}\Sigma_{1}^{\infty}(-)^{t-1}2^{2t-2}E'_{2t-2}(n)q^{n}\frac{x^{2t-2}}{(2t-2)!},$$

$$k'\rho\,\operatorname{nd}\rho x = 1+4\Sigma_{1}^{\infty}\Sigma_{1}^{\infty}(-)^{t-1+n}2^{2t-2}E'_{2t-2}(n)q^{n}\frac{x^{2t-2}}{(2t-2)!};$$

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$$\rho\,\mathrm{ns}\,\rho x = \operatorname{cec} x + 4\Sigma_1^\infty \Sigma_1^\infty (-)^{t-1} \Delta_{2t-1}(n) q^n \frac{x^{2t-1}}{(2t-1)!},$$

$$\rho\,\mathrm{ds}\,\rho x = \operatorname{cec} x + 4\Sigma_1^\infty \Sigma_1^\infty (-)^{t-1+n} \Delta_{2t-1}(n) q^n \frac{x^{2t-1}}{(2t-1)!},$$

$$\rho\,\mathrm{dc}\,\rho x = \sec x + 4\Sigma_1^\infty \Sigma_1^\infty (-)^{t-1} E_{2t-2}(n) q^n \frac{x^{2t-2}}{(2t-2)!},$$

$$k'\rho\,\mathrm{nc}\,\rho x = \sec x + 4\Sigma_1^\infty \Sigma_1^\infty (-)^{t-1+n} E_{2t-2}(n) q^n \frac{x^{2t-2}}{(2t-2)!};$$

$$\rho\,\mathrm{zs}\,\rho x = \cot x + 4\Sigma_1^\infty \Sigma_1^\infty (-)^{t-1} 2^{2t-1} \sigma_{2t-1}(n) q^{2n} \frac{x^{2t-1}}{(2t-1)!},$$

$$\rho\,\mathrm{cs}\,\rho x = \cot x - 4\Sigma_1^\infty \Sigma_1^\infty (-)^{t-1} 2^{2t-1} \zeta'_{2t-1}(x) q^{2n} \frac{x^{2t-1}}{(2t-1)!},$$

$$\rho\,\mathrm{zc}\,\rho x = -\tan x - 4\Sigma_1^\infty \Sigma_1^\infty (-)^{t-1} 2^{2t-1} \zeta'_{2t-1}(n) q^{2n} \frac{x^{2t-1}}{(2t-1)!},$$

$$k'\rho\,\mathrm{sc}\,\rho x = \tan x - 4\Sigma_1^\infty \Sigma_1^\infty (-)^{t-1} 2^{2t-1} \xi_{2t-1}(n) q^{2n} \frac{x^{2t-1}}{(2t-1)!},$$

In these formulæ n denotes any positive integral number, and m any positive uneven integral number. As in § 7 the letter M denotes $\frac{1}{2}(m-1)$. The letter t is to have all integral values from 1 to ∞.

The functions $\sigma_t(n)$, $\zeta_t(n)$, &c., §§ 53, 54.

§ 53. The definitions of the coefficients $\Delta_t(n)$, $E_t(n)$, &c., expressed in the notation of § 4, are:

$$\sigma_t(n) = \Sigma_n d^t, \qquad \xi_t(n) = \Sigma_n (-)^{d+d'} d^t,$$
$$\zeta_t(n) = \Sigma_n (-)^{d-1} d^t, \qquad \zeta'_t(n) = \Sigma_n (-)^{d'-1} d^t,$$
$$\Delta_t(n) = \Sigma_n \delta^t, \qquad \Delta'_t(n) = \Sigma_n \delta'^t,$$
$$E_t(n) = \Sigma_n (-)^{\frac{1}{2}(\delta-1)} \delta^t, \qquad E'_t(n) = \Sigma_n (-)^{\frac{1}{2}(\delta-1)} \delta'^t.$$

These definitions may be expressed in words as follows:

$\sigma_t(n) =$ the sum of the t^{th} powers of all the divisors of n,

$\zeta_t(n) =$ { the sum of the t^{th} powers of the uneven divisors of n,
− ..... even .....

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$$\zeta_t'(n) = \begin{cases} \text{the sum of the } t^{\text{th}} \text{ powers of the divisors of } n \text{ whose conjugates are uneven,} \\ - \ldots\ldots \text{ even,} \end{cases}$$

$$\zeta_t(n) = \begin{cases} \text{the sum of the } t^{\text{th}} \text{ powers of the uneven divisors of } n \text{ whose conjugates are uneven,} \\ + \text{the sum of the } t^{\text{th}} \text{ powers of the even divisors of } n \text{ whose conjugates are even,} \\ - \text{the sum of the } t^{\text{th}} \text{ powers of the uneven divisors of } n \text{ whose conjugates are even,} \\ - \text{the sum of the } t^{\text{th}} \text{ powers of the even divisors of } n \text{ whose conjugates are uneven,} \end{cases}$$

$$\Delta_t(n) = \text{the sum of the } t^{\text{th}} \text{ powers of the uneven divisors of } n,$$

$$E_t(n) = \begin{cases} \text{the sum of the } t^{\text{th}} \text{ powers of the divisors of } n \text{ of the form } 4p+1, \\ - \ldots\ldots 4p+3, \end{cases}$$

$$\Delta_t'(n) = \text{the sum of the } t^{\text{th}} \text{ powers of those divisors of } n \text{ whose conjugates are uneven,}$$

$$E_t'(n) = \begin{cases} \text{the sum of the } t^{\text{th}} \text{ powers of those divisors of } n \text{ whose conjugates are of the form } 4p+1, \\ - \ldots\ldots 4p+3, \end{cases}$$

If n be an even number $= 2^u m$, where m is uneven, we have

$$\Delta_t'(n) = 2^{ut}\Delta_t(n),$$

$$\sigma_t(n) = \frac{2^{(u+1)t}-1}{2^t-1}\Delta_t(n),$$

$$\zeta_t(n) = -\frac{2^{(u+1)t}-2^{t+1}+1}{2^t-1}\Delta_t(n),$$

$$\zeta_t'(n) = \frac{2^{(u+1)t}-2^{u+1}+1}{2^t-1}\Delta_t(n),$$

$$\xi_t(n) = -\frac{2^{(u+1)t}-2^{u+1}+2^{t+1}-1}{2^t-1}\Delta_t(n).$$

The first four equations are also true when $s=0$, but in the last equation s must be >0. Corresponding to $s=0$, the functions all become equal, viz. m being uneven,

$$\Delta_t'(m) = \sigma_t(m) = \zeta_t(m) = \zeta_t'(m) = \xi_t(m).$$

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If $n=2^s m=2^s a^\alpha b^\beta c^\gamma\ldots$, where a, b, c, ... are any prime numbers different from one another, then

$$\Delta_t(n)=\frac{(a^{\alpha t+t}-1)\,(b^{\beta t+t}-1)\,(c^{\gamma t+t}-1)\ldots}{(a^t-1)\,(b^t-1)\,(c^t-1)\ldots}$$

If, therefore, f_t denote any one of the functions Δ'_t, σ_t, ζ_t, ζ'_t, ξ_t, then

$$f_t(n)=f_t(2^s)\,\Delta_t(a^\alpha)\,\Delta_t(b^\beta)\,\Delta_t(c^\gamma)\ldots;$$

and, if $n=2^s m_1 m_2 m_3\ldots$, where m_1, m_2, m_3, ... are any uneven numbers which are all prime to each other,

$$f_t(n)=f_t(2^s)\,\Delta_t(m_1)\,\Delta_t(m_2)\,\Delta_t(m_3)\ldots.$$

Even more generally, if $n=n_1 n_2 n_3\ldots$, where n_1, n_2, n_3, ... are any numbers which are all prime to each other,

$$f_t(n)=f_t(n_1)f_t(n_2)f_t(n_3)\ldots.$$

§ 54. If $D_t(n)$ and $D'_t(n)$ denote respectively the sum of the even divisors of n, and the sum of those divisors of n whose conjugates are even, we have evidently

$$\begin{aligned}
\sigma_t(n)&=\Delta_t(n)+D_t(n)=\Delta'_t(n)+D'_t(n),\\
\zeta_t(n)&=\Delta_t(n)-D_t(n),\\
\zeta'_t(n)&=\Delta'_t(n)-D'_t(n),\\
\xi_t(n)&=-3\sigma_t(n)+2D_t(n)+2D'_t(n).
\end{aligned}$$

and, if as before $n=2^s m$,

$$D_t(n)=\frac{2^{st+t}-2^t}{2^t-1}\,\Delta_t(n),\qquad D'_t(n)=\frac{2^{st}-1}{2^t-1}\,\Delta_t(n).$$

We may express all the functions very conveniently in terms of $\sigma_t(n)$ and $D'_t(n)$. We thus find

$$\begin{aligned}
\Delta_t(n)&=\sigma_t(n)-2^t D'_t(n),\\
D_t(n)&=2^t D'_t(n),\\
\Delta'_t(n)&=\sigma_t(n)-D'_t(n),\\
D'_t(n)&=D'_t(n),\\
\zeta_t(n)&=\sigma_t(n)-2^{t+1}D'_t(n),\\
\zeta'_t(n)&=\sigma_t(n)-2D'_t(n),\\
\xi_t(n)&=-3\sigma_t(n)+(2^{t+1}+2)\,D'_t(n).*
\end{aligned}$$

* With the exception of the formulæ relating to $\xi_t(n)$, the results contained in §§ 53 and 54 were given in *Messenger*, vol. XV., pp. 1–5 (May, 1885).

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It is easy to see that $D'_t(2n) = \sigma_t(n)$. If, therefore, we define $\sigma_t(\frac{1}{2}n)$ to denote zero when n is uneven, $D'_t(n) = \sigma_t(\frac{1}{2}n)$, and we may express all the functions very conveniently in terms of σ_t by the formulæ

$$\Delta_t(n) = \sigma_t(n) - 2^t\sigma_t(\tfrac{1}{2}n),$$
$$D_t(n) = 2^t\sigma_t(\tfrac{1}{2}n),$$
&c., &c.

§ 55. In the special case of $t = 1$ the values of the functions ζ' and Δ become equal, viz., $\zeta_1'(n) = \Delta_1(n)$. We have, also, $\xi_1(n) = \{(-)^{n-1}2 - 1\}\Delta_1(n)$, so that $\xi_1(n) = \Delta_1(n)$ if n is uneven, and $= -3\Delta_1(n)$ if n is even.

It is convenient to omit the suffix when it is unity, and write $\sigma(n)$, $\Delta(n)$, &c. instead of $\sigma_1(n)$, $\Delta_1(n)$, &c.

The functions $E_t(n)$ and $E'_t(n)$, §§ 56, 57.

§ 56. The functions $E_t(n)$ and $E'_t(n)$ may be calculated by formulæ of the same kind as those given in § 53.

For, if $n = 2^\varepsilon m = 2^\varepsilon e^\varepsilon f^\phi \ldots u^\upsilon v^\eta \ldots$, where $e, f, \ldots$ are different primes of the form $4r+3$, and $u, v, \ldots$ are different primes of the form $4r+1$, it can be shown that

$$E_t(n) = (-)^{\varepsilon+\phi+\ldots} \frac{e^{t\varepsilon+t} + (-1)^t}{e^t+1} \frac{f^{t\phi+t} + (-1)^\phi}{f^t+1} \ldots \frac{u^{t\upsilon+t}-1}{u^t-1} \frac{v^{t\eta+t}-1}{v^t-1} \ldots .$$

Thus $E_t(n)$ is positive or negative according as $\varepsilon + \phi + \ldots$ is even or uneven, that is, according as m, the greatest uneven divisor of n, is of the form $4r+1$ or $4r+3$.

It can be shown that $E'_t(m) = (-)^{\frac{1}{2}(m-1)}E_t(m)$, and that, if $n = 2^\alpha m$,

$$E'_t(n) = 2^\alpha E'_t(m) = (-)^{\frac{1}{2}(m-1)}2^\alpha E_t(m) = (-)^{\frac{1}{2}(m-1)}2^\alpha E_t(n).$$

If $n = 2^\alpha a^\alpha b^\beta c^\gamma \ldots$ where $a, b, c, \ldots$ are any uneven primes different from one another,

$$E_t(n) = E_t(a^\alpha)\, E_t(b^\beta)\, E_t(c^\gamma) \ldots,$$

and, if $n = n_1 n_2 n_3 \ldots$ where $n_1, n_2, n_3, \ldots$ are any numbers which are all prime to each other (except that they may contain any power of 2), then

$$E_t(n) = E_t(n_1)\, E_t(n_2)\, E_t(n_3) \ldots,$$

and

$$E'_t(n) = E'_t(n_1)\, E'_t(n_2)\, E'_t(n_3) \ldots .$$

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§ 57. When $t=0$ the functions E and E' become equal to one another, viz., $E_0'(n)=E_0(n)$. It can be shown that $E_0(m)$ is zero if m is of the form $4r+3$, and that in general $E_0(n)$ is zero unless the greatest uneven divisor of n is of the form $4r+1$, or, in other words, unless all the prime factors of n which are of the form $4r+3$ have even exponents.

Remarks on the expansions in powers of q and x, § 58.

§ 58. In the formulæ of § 52 it will be noticed that the suffixes of the functions σ, ζ, ζ', ξ, Δ, Δ' are uneven, and that those of the functions E and E' are even. In four of the sixteen expansions (viz. those involving σ, ζ, ζ', ξ) the coefficients depend upon the even as well as the uneven divisors of the exponents of q (or of its half or double), while in twelve they depend only upon uneven divisors or their conjugates. In the first two formulæ of the system we may replace $\Delta_{2t-1}(m)$ by $\sigma_{2t-1}(m)$; the Δ-form, however, seems preferable, as the divisors are uneven.

With respect to the calculation of the coefficients σ_t, E_t, &c., it is evident from §§ 53–57 that we may readily form a table of the values of any one of these functions, $\phi_t(n)$, by the very simple process of first writing down the values of $\phi_t(n)$ for prime numbers, and then deducing from them the values of $\phi_t(n)$ for composite numbers by simple multiplication.*

Values of sixteen double summations, §§ 59–67.

§ 59. From § 49 we have

$$\rho\,\mathrm{zs}\,\rho x=\Sigma_{r_2=-\infty}^{r_2=\infty}\Sigma_{s_2=-\infty}^{s_2=\infty}\frac{1}{x+\frac{1}{2}r_2\pi+\frac{1}{2}s_2 i\mu}.$$

Expanding the expression $(x+\frac{1}{2}r_2\pi+\frac{1}{2}s_2 i\mu)^{-1}$ in ascending powers of x, we find

$$\rho\,\mathrm{zs}\,\rho x=-\Sigma_{t=1}^{t=\infty}\Sigma_{r_2=-\infty}^{r_2=\infty}\Sigma_{s_2=-\infty}^{s_2=\infty}\frac{2^{2t}x^{2t-1}}{(r_2\pi+s_2 i\mu)^{2t}}.$$

* A table of the values of $\sigma(n)$, $E_0(n)$, $E_2(n)$ as far as $n=100$ was given in the *Quarterly Journal*, vol. xx., pp. 164, 165 (1884), and a table of $E_0(n)$ as far as $n=1000$ in the *Proc. Lond. Math. Soc.*, vol. xv., p. 106 (1884). In my *Lectures on Elliptic Functions* (pp. 257, 264, 277) I have given tables of $\Delta(n)$, $E_2(n)$, $E_2'(n)$, $\Delta_2'(n)$, $\zeta_3(n)$ as far as $n=100$. I have also formed a table of $\sigma(n)$ as far as $n=10{,}000$; this table, which was printed and stereotyped in 1885, has not yet been published.

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Now, from § 27,

$$\rho\,\mathrm{zs}\,\rho x = \frac{1}{x} - \tfrac{1}{3}\rho^2(\mathrm{i}+\mathrm{g}+\mathrm{e})\,x - \Sigma_1^\infty \frac{F_{t+1}(1,k^2)\,\rho^{2t+2}x^{2t+1}}{(2t+1)!}.$$

Equating the coefficients of x^{2t-1}, we find, if $t>1$,

$$\Sigma_{r_2=-\infty}^{r_2=\infty}\Sigma_{s_2=-\infty}^{s_2=\infty}\frac{2^{2t}}{(r_2\pi+s_2 i\mu)^{2t}} = \frac{F_t(1,k^2)\,\rho^{2t}}{(2t-1)!}.$$

We may attach the i to π instead of to μ by multiplying the numerator and denominator by i^{2t} and changing the sign of i. We thus obtain the formula

$$\Sigma_{s_2=-\infty}^{s_2=\infty}\Sigma_{r_2=-\infty}^{r_2=\infty}\frac{2^{2t}}{(s_2\mu+r_2 i\pi)^{2t}} = (-)^t\frac{F_t(1,k^2)\,\rho^{2t}}{(2t-1)!}$$

$$= \frac{F_t(1,k'^2)\,\rho^{2t}}{(2t-1)!} = \frac{F_t(k^2,-k'^2)\,\rho^{2t}}{(2t-1)!} \quad (\S\,29).$$

The value of the double summation is independent of the ratio of the r-limits to the s-limits.

§ 60. If $t=1$, we have

$$\Sigma_{r_2=-\infty}^{r_2=\infty}\Sigma_{s_2=-\infty}^{s_2=\infty}\frac{4}{(r_2\pi+s_2 i\mu)^2} = \tfrac{1}{3}\frac{4K(I+G+E)}{\pi^2},$$

viz. $$\Sigma_{s_2=-\infty}^{s_2=\infty}\Sigma_{r_2=-\infty}^{r_2=\infty}\frac{4}{(s_2\mu+r_2 i\pi)^2} = -\tfrac{1}{3}\frac{4K(I+G+E)}{\pi^2}.$$

In this equation the r-limits are supposed to be infinitely larger than the s-limits. If the s-limits are infinitely larger than the r-limits, the formula for $\rho\,\mathrm{zs}\,\rho x$ is (§ 49, p. 43)

$$\rho\,\mathrm{zs}\,\rho x = \Sigma_{-\infty}^{\infty}\Sigma_{-\infty}^{\infty}\frac{1}{x+\frac{1}{2}r_2\pi+\frac{1}{2}s_2 i\mu} - \frac{2x}{\mu},$$

and, by equating coefficients, we find in this case

$$\Sigma_{s_2=-\infty}^{s_2=\infty}\Sigma_{r_2=-\infty}^{r_2=\infty}\frac{4}{(s_2\mu+r_2 i\pi)^2} = \frac{2}{\mu} - \tfrac{1}{3}\frac{4K(I+G+E)}{\pi^2}$$

$$= \frac{4K^2(I'+G'+E')}{3\pi^2 K'}.$$

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This formula is readily deducible from the previous result, in which the r-limits were infinitely greater than the s-limits; for, putting $\frac{\pi^2}{\mu}$ for μ (which corresponds to the change of modulus from k to k') in the first equation of this section, we find, the r-limits being infinitely larger than the s-limits,

$$\Sigma_{r_2=-\infty}^{r_2=\infty} \Sigma_{s_2=-\infty}^{s_2=\infty}{}' \frac{4}{(r_2\mu + s_2 i\pi)^2} = \tfrac{1}{8} \frac{4K'(I' + G' + E')}{\mu^2},$$

which is equivalent to the second result.

§ 61. By expanding the sixteen series in § 49, and equating the coefficients of x^{2t} or x^{2t-1} to those of the series derived from § 27, we obtain the following system of formulæ giving the values of the sixteen double summations.

In the first four series (*i.e.* the first group) the simultaneous pair of values $r_2 = 0$, $s_2 = 0$ is excluded.

$$\Sigma_{-\infty}^{\infty}\Sigma_{-\infty}^{\infty} \frac{2^{2t}}{(s_2\mu + r_2 i\pi)^{2t}} = F_t(k^2, -k'^2) \frac{\rho^{2t}}{(2t-1)!},$$

$$\Sigma_{-\infty}^{\infty}\Sigma_{-\infty}^{\infty}(-)^r \frac{2^{2t}}{(s_2\mu + r_2 i\pi)^{2t}} = (-)^{t-1}\chi_t(1, k^2) \frac{\rho^{2t}}{(2t-1)!},$$

$$-\Sigma_{-\infty}^{\infty}\Sigma_{-\infty}^{\infty}(-)^s \frac{2^{2t}}{(s_2\mu + r_2 i\pi)^{2t}} = \chi_t(1, k'^2) \frac{\rho^{2t}}{(2t-1)!},$$

$$-\Sigma_{-\infty}^{\infty}\Sigma_{-\infty}^{\infty}(-)^{r+s} \frac{2^{2t}}{(s_2\mu + r_2 i\pi)^{2t}} = \chi_t(k^2, -k'^2) \frac{\rho^{2t}}{(2t-1)!};$$

$$-\Sigma_{-\infty}^{\infty}\Sigma_{-\infty}^{\infty} \frac{2^{2t}}{(s_2\mu + r_1 i\pi)^{2t}} = (-)^{t-1}k'^2 f_{t-2}^{\,2}(1, k'^2) \frac{\rho^{2t}}{(2t-1)!},$$

$$-i\Sigma_{-\infty}^{\infty}\Sigma_{-\infty}^{\infty}(-)^r \frac{2^{2t+1}}{(s_2\mu + r_1 i\pi)^{2t+1}} = (-)^t k'^2 \psi_{t-1}(k'^2, 1) \frac{\rho^{2t+1}}{(2t)!},$$

$$\Sigma_{-\infty}^{\infty}\Sigma_{-\infty}^{\infty}(-)^s \frac{2^{2t}}{(s_2\mu + r_1 i\pi)^{2t}} = (-)^t k' \phi_{t-1}(1, k'^2) \frac{\rho^{2t}}{(2t-1)!},$$

$$i\Sigma_{-\infty}^{\infty}\Sigma_{-\infty}^{\infty}(-)^{r+s} \frac{2^{2t+1}}{(s_2\mu + r_1 i\pi)^{2t+1}} = (-)^{t-1}k' \psi_{t-1}(1, k'^2) \frac{\rho^{2t+1}}{(2t)!};$$

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$$\Sigma_{-\infty}^{\infty}\Sigma_{-\infty}^{\infty}\frac{2^{2t}}{(s_1\mu+r_2i\pi)^{2t}} = k^2 f_{t-2}(1,k^2)\frac{\rho^{2t}}{(2t-1)!},$$

$$\Sigma_{-\infty}^{\infty}\Sigma_{-\infty}^{\infty}(-)^r\frac{2^{2t}}{(s_1\mu+r_2i\pi)^{2t}} = k\phi_{t-1}(1,k^2)\frac{\rho^{2t}}{(2t-1)!},$$

$$-\Sigma_{-\infty}^{\infty}\Sigma_{-\infty}^{\infty}(-)^s\frac{2^{2t+1}}{(s_1\mu+r_2i\pi)^{2t+1}} = k^2\psi_{t-1}(k^2,1)\frac{\rho^{2t+1}}{(2t)!},$$

$$-\Sigma_{-\infty}^{\infty}\Sigma_{-\infty}^{\infty}(-)^{r+s}\frac{2^{2t+1}}{(s_1\mu+r_2i\pi)^{2t+1}} = k\psi_{t-1}(1,k^2)\frac{\rho^{2t+1}}{(2t)!};$$

$$-\Sigma_{-\infty}^{\infty}\Sigma_{-\infty}^{\infty}\frac{2^{2t}}{(s_1\mu+r_1i\pi)^{2t}} = k^2k'^2 f_{t-2}(k'^2,-k^2)\frac{\rho^{2t}}{(2t-1)!},$$

$$-i\Sigma_{-\infty}^{\infty}\Sigma_{-\infty}^{\infty}(-)^r\frac{2^{2t+1}}{(s_1\mu+r_1i\pi)^{2t+1}} = kk'^2\psi_{t-1}(k'^2,-k^2)\frac{\rho^{2t+1}}{(2t)!},$$

$$\Sigma_{-\infty}^{\infty}\Sigma_{-\infty}^{\infty}(-)^s\frac{2^{2t+1}}{(s_1\mu+r_1i\pi)^{2t+1}} = (-)^{t-1}k^2k'\psi_{t-1}(k^2,-k'^2)\frac{\rho^{2t+1}}{(2t)!},$$

$$-i\Sigma_{-\infty}^{\infty}\Sigma_{-\infty}^{\infty}(-)^{r+s}\frac{2^{2t}}{(s_1\mu+r_1i\pi)^{2t}} = kk'\phi_{t-1}(k'^2,-k^2)\frac{\rho^{2t}}{(2t-1)!}.$$

In the above formulæ the notation is the same as in §§ 37–49, viz., s_2 and r_2 denote even integers and s_1 and r_1 uneven integers. The exponent s denotes $\frac{1}{2}s_2$ or $\frac{1}{2}(s_1+1)$ according as s_2 or s_1 occurs in the expression subject to the double summation, and similarly r denotes $\frac{1}{2}r_2$ or $\frac{1}{2}(r_1+1)$; or, in other words, we may suppose $s_2=2s$, $s_1=2s-1$, $r_2=2r$, $r_1=2r-1$.

The integer t may be put equal to unity in all the series except the four which are the first of their groups (*i.e.* those derived from the Zeta series, in which all the terms are positive). Excluding the case $t=1$ of these four series, all the results are independent of the ratio of the r-limits to the s-limits.

§ 62. Proceeding as in § 60 we find that, if the r-limits be indefinitely larger than the s-limits,

$$\Sigma_{-\infty}^{\infty}\Sigma_{-\infty}^{\infty}\frac{4}{(s_2\mu+r_2i\pi)^2} = -\frac{1}{3}\frac{4K(I+G+E)}{\pi^2},$$

$$\Sigma_{-\infty}^{\infty}\Sigma_{-\infty}^{\infty}\frac{4}{(s_2\mu+r_1i\pi)^2} = -\frac{4KE}{\pi^2},$$

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$$\Sigma_{-\infty}^{\infty}\Sigma_{-\infty}^{\infty}\frac{4}{(s_1\mu+r_2i\pi)^2}=-\frac{4KI}{\pi^2},$$

$$\Sigma_{-\infty}^{\infty}\Sigma_{-\infty}^{\infty}\frac{4}{(s_1\mu+r_1i\pi)^2}=-\frac{4KG}{\pi^2};$$

and that, if the s-limits be infinitely larger than the r-limits, these four double summations are respectively equal to

$$\tfrac{1}{8}\frac{4K'(I'+G'+E')}{\mu^2},\quad \frac{4K'I'}{\mu^2},\quad \frac{4K'E'}{\mu^2},\quad \frac{4K'G'}{\mu^2}.$$

§ 63. Corresponding to the value $s_2=0$, each of the first eight double summations in § 61 contains a single series of terms which are independent of μ.

Taking the first double summation we have

$$\Sigma_{-\infty}^{\infty}\Sigma_{-\infty}^{\infty}\frac{2^{2t}}{(s_2\mu+r_2i\pi)^{2t}}=2\Sigma_{s_2=2}^{s_2=\infty}\Sigma_{-\infty}^{\infty}\frac{2^{2t}}{(s_2\mu+r_2i\pi)^{2t}}+\Sigma_{-\infty}^{\infty}\frac{2^{2t}}{(r_2i\pi)^{2t}};$$

and, denoting by B_t the t^{th} Bernoullian number,

$$\Sigma_{-\infty}^{\infty}\frac{2^{2t}}{(r_2i\pi)^{2t}}=(-)^t\frac{1}{\pi^{2t}}2\Sigma_1^{\infty}\frac{1}{r^{2t}}=(-)^t\frac{1}{\pi^{2t}}\frac{(2\pi)^{2t}B_t}{(2t)!}$$

$$=(-)^t\frac{2^{2t}B_t}{(2t)!}=\frac{F_t(1,1)}{(2t-1)!}=\frac{F_t(0,-1)}{(2t-1)!}\quad(\S\,31),$$

whence

$$2\Sigma_{s_2=2}^{s_2=\infty}\Sigma_{-\infty}^{\infty}\frac{2^{2t}}{(s_2\mu+r_2i\pi)^{2t}}=\frac{F_t(k^2,-k'^2)\,\rho^{2t}-F_t(0,-1)}{(2t-1)!}$$

$$=\frac{F_t(1,k'^2)\,\rho^{2t}-F_t(1,1)}{(2t-1)!}.$$

§ 64. We may therefore express the system of results given in § 61 in the following form, in which s_2 receives all positive even values from 2 to ∞, and s_1 all positive uneven values from 1 to ∞. The limits of r_2 and r_1 are $\pm\infty$, as before.*

* The formulæ in this section may be obtained directly by equating coefficients in the series for ρ ns ρx − cec x, &c., as in § 69.

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$$2^{2t+1}\ \Sigma_2^\infty \Sigma_{-\infty}^\infty \frac{1}{(s_2\mu + r_2 i\pi)^{2t}} = \frac{F_t(k^2, -k'^2)\,\rho^{2t} - F_t(0,-1)}{(2t-1)!},$$

$$2^{2t+1}\ \Sigma_2^\infty \Sigma_{-\infty}^\infty \frac{(-1)^{\frac{1}{2}r_2}}{(s_2\mu + r_2 i\pi)^{2t}} = (-)^{t-1}\frac{\chi_t(1, k^2)\,\rho^{2t} - \chi_t(1,0)}{(2t-1)!},$$

$$-2^{2t+1}\ \Sigma_2^\infty \Sigma_{-\infty}^\infty \frac{(-1)^{\frac{1}{2}s_2}}{(s_2\mu + r_2 i\pi)^{2t}} = \frac{\chi_t(1, k'^2)\,\rho^{2t} - \chi_t(1,1)}{(2t-1)!},$$

$$-2^{2t+1}\ \Sigma_2^\infty \Sigma_{-\infty}^\infty \frac{(-1)^{\frac{1}{2}(r_2+s_2)}}{(s_2\mu + r_2 i\pi)^{2t}} = \frac{\chi_t(k^2, -k'^2)\,\rho^{2t} - \chi_t(0,-1)}{(2t-1)!};$$

$$-2^{2t+1}\ \Sigma_2^\infty \Sigma_{-\infty}^\infty \frac{1}{(s_2\mu + r_1 i\pi)^{2t}} = (-)^{t-1}\frac{k'^2 f_{t-2}(1, k'^2)\,\rho^{2t} - f_{t-2}(1,1)}{(2t-1)!},$$

$$2^{2t+2} i\,\Sigma_2^\infty \Sigma_{-\infty}^\infty \frac{(-1)^{\frac{1}{2}(r_1-1)}}{(s_2\mu + r_1 i\pi)^{2t+1}} = (-)^t\ \frac{k'^2\psi_{t-1}(k'^2, 1)\rho^{2t+1} - \psi_{t-1}(1,1)}{(2t)!},$$

$$2^{2t+1}\ \Sigma_2^\infty \Sigma_{-\infty}^\infty \frac{(-1)^{\frac{1}{2}s_2}}{(s_2\mu + r_1 i\pi)^{2t}} = (-)^t\ \frac{k'\phi_{t-1}(1, k'^2)\,\rho^{2t} - \phi_{t-1}(1,1)}{(2t-1)!},$$

$$-2^{2t+2} i\,\Sigma_2^\infty \Sigma_{-\infty}^\infty \frac{(-1)^{\frac{1}{2}(s_2+r_1-1)}}{(s_2\mu + r_1 i\pi)^{2t+1}} = (-)^{t-1}\frac{k'\psi_{t-1}(1, k'^2)\rho^{2t+1} - \psi_{t-1}(1,1)}{(2t)!};$$

$$2^{2t+1}\ \Sigma_1^\infty \Sigma_{-\infty}^\infty \frac{1}{(s_1\mu + r_2 i\pi)^{2t}} = \frac{k^2 f_{t-2}(1, k^2)\,\rho^{2t}}{(2t-1)!},$$

$$2^{2t+1}\ \Sigma_1^\infty \Sigma_{-\infty}^\infty \frac{(-1)^{\frac{1}{2}r_2}}{(s_1\mu + r_2 i\pi)^{2t}} = \frac{k\phi_{t-1}(1, k^2)\,\rho^{2t}}{(2t-1)!},$$

$$2^{2t+2}\ \Sigma_1^\infty \Sigma_{-\infty}^\infty \frac{(-1)^{\frac{1}{2}(s_1-1)}}{(s_1\mu + r_2 i\pi)^{2t+1}} = \frac{k^2\psi_{t-1}(k^2, 1)\,\rho^{2t+1}}{(2t)!},$$

$$2^{2t+2}\ \Sigma_1^\infty \Sigma_{-\infty}^\infty \frac{(-1)^{\frac{1}{2}(r_2+s_1-1)}}{(s_1\mu + r_2 i\pi)^{2t+1}} = \frac{k\psi_{t-1}(1, k^2)\,\rho^{2t+1}}{(2t)!};$$

$$-2^{2t+1}\ \Sigma_1^\infty \Sigma_{-\infty}^\infty \frac{1}{(s_1\mu + r_1 i\pi)^{2t}} = \frac{k^2 k'^2 f_{t-2}(k'^2, -k^2)\,\rho^{2t}}{(2t-1)!},$$

$$2^{2t+2} i\,\Sigma_1^\infty \Sigma_{-\infty}^\infty \frac{(-1)^{\frac{1}{2}(r_1-1)}}{(s_1\mu + r_1 i\pi)^{2t+1}} = \frac{kk'^2\psi_{t-1}(k'^2, -k^2)\,\rho^{2t+1}}{(2t)!},$$

$$-2^{2t+2}\ \Sigma_1^\infty \Sigma_{-\infty}^\infty \frac{(-1)^{\frac{1}{2}(s_1-1)}}{(s_1\mu + r_1 i\pi)^{2t+1}} = (-)^{t-1}\frac{k^2 k'\psi_{t-1}(k^2, -k'^2)\,\rho^{2t+1}}{(2t)!},$$

$$2^{2t+1} i\,\Sigma_1^\infty \Sigma_{-\infty}^\infty \frac{(-1)^{\frac{1}{2}(r_1+s_1)}}{(s_1\mu + r_1 i\pi)^{2t}} = \frac{kk'\phi_{t-1}(k'^2, -k^2)\,\rho^{2t}}{(2t-1)!}.$$

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The remarks at the end of § 61 relating to the admissible values of t and the ratio of the r-limits to the s-limits apply of course to these formulæ also.

§ 65. Corresponding to § 62 we have, if the r-limits are infinitely larger than the s-limits,

$$8\sum_{s_2=2}^{s_2=\infty}\sum_{-\infty}^{\infty}\frac{1}{(s_2\mu+r_2i\pi)^2}=\tfrac{1}{8}-\tfrac{1}{8}\frac{4K(I+G+E)}{\pi^2},$$

$$8\sum_{s_2=2}^{s_2=\infty}\sum_{-\infty}^{\infty}\frac{1}{(s_2\mu+r_1i\pi)^2}=1-\frac{4KE}{\pi^2},$$

$$8\sum_{s_1=1}^{s_1=\infty}\sum_{-\infty}^{\infty}\frac{1}{(s_1\mu+r_2i\pi)^2}=-\frac{4KI}{\pi^2},$$

$$8\sum_{s_1=1}^{s_1=\infty}\sum_{-\infty}^{\infty}\frac{1}{(s_1\mu+r_1i\pi)^2}=-\frac{4KG}{\pi^2}.$$

If the s-limits are infinitely larger than the r-limits, the values of these double summations are respectively

$$\tfrac{1}{8}+\tfrac{1}{8}\frac{4K'(I'+G'+E')}{\mu^2},\quad 1+\frac{4K'I'}{\mu^2},\quad \frac{4K'E'}{\mu^2},\quad \frac{4K'G'}{\mu^2}.$$

§ 66. The results which correspond to the case $t=0$ of the summations in which the exponent is $2t+1$ may be deduced at sight from § 49 (p. 42) by putting $x=0$ in the formulæ for the six even elliptic functions. In the equations so obtained it is supposed that the positive and negative infinite limits are exactly equal in magnitude, and that the r-limits are infinitely larger than the s-limits. In two of the equations the term ± 1 occurs, the sign depending upon the form of the s_1-limits. By substituting $\frac{\pi^2}{\mu}$ for μ in these formulæ we may deduce, as in § 60, the values of the series when the s-limits are infinitely larger than the r-limits.

The results in the two cases are given on the next page. As in § 42, v_1, v_2 denote the positive limiting values of s_1, s_2; and u_1, u_2 are used to denote the positive limiting values of r_1, r_2. In the first column of results u_1, u_2 are supposed to be infinitely larger than v_1, v_2, and in the second column infinitely smaller.

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$$2i\,\Sigma_{-\infty}^{\infty}\Sigma_{-\infty}^{\infty}\frac{(-1)^{\frac{1}{2}(r_1-1)}}{s_2\mu+r_1 i\pi} = \rho, \qquad (u>v) \qquad = \rho+(-)^{\frac{1}{2}(u_1-1)}\frac{\pi}{\mu}, \qquad (v>u)$$

$$2i\,\Sigma_{-\infty}^{\infty}\Sigma_{-\infty}^{\infty}\frac{(-1)^{\frac{1}{2}(s_2+r_1-1)}}{s_2\mu+r_1 i\pi} = k'\rho, \qquad = k'\rho,$$

$$2\,\Sigma_{-\infty}^{\infty}\Sigma_{-\infty}^{\infty}\frac{(-1)^{\frac{1}{2}(s_1-1)}}{s_1\mu+r_2 i\pi} = \rho+(-1)^{\frac{1}{2}(v_1-1)}, \qquad = \rho,$$

$$2\,\Sigma_{-\infty}^{\infty}\Sigma_{-\infty}^{\infty}\frac{(-1)^{\frac{1}{2}(r_2+s_1-1)}}{s_1\mu+r_2 i\pi} = k\rho, \qquad = k\rho,$$

$$2i\,\Sigma_{-\infty}^{\infty}\Sigma_{-\infty}^{\infty}\frac{(-1)^{\frac{1}{2}(r_1-1)}}{s_1\mu+r_1 i\pi} = k\rho, \qquad = k\rho+(-)^{\frac{1}{2}(u_1-1)}\frac{\pi}{\mu},$$

$$2\,\Sigma_{-\infty}^{\infty}\Sigma_{-\infty}^{\infty}\frac{(-1)^{\frac{1}{2}(s_1-1)}}{s_1\mu+r_1 i\pi} = k'\rho+(-1)^{\frac{1}{2}(v_1-1)}, \qquad = k'\rho.$$

The first two double summations may be written in the form

$$1+4i\,\Sigma_{s_2=2}^{s_2=\infty}\Sigma_{-\infty}^{\infty}\frac{(-1)^{\frac{1}{2}(r_1-1)}}{s_2\mu+r_1 i\pi},$$

$$1+4i\,\Sigma_{s_2=2}^{s_2=\infty}\Sigma_{-\infty}^{\infty}\frac{(-1)^{\frac{1}{2}(s_2+r_1-1)}}{s_2\mu+r_1 i\pi}.$$

In the last four we may replace $2\,\Sigma_{-\infty}^{\infty}\Sigma_{-\infty}^{\infty}$ by $4\Sigma_{s_1=1}^{s_1=\infty}\Sigma_{-\infty}^{\infty}$.

We may also obtain the preceding system of results by putting $x=0$ in the formulæ of § 42; for example, from the first formula on p. 36 we have, if u be infinitely larger than v,

$$\rho-(-1)^s = -2i\,\Sigma_{-v_1}^{v_1}\Sigma_{-\infty}^{\infty}(-)^s\frac{1}{r_2\pi+s_1 i\mu},$$

which is equivalent to the first equality in the third of the above results. An independent method of proving these formulæ is given in § 77.

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§ 67. On putting for μ its value $\frac{\pi K'}{K}$ in § 61, the factor ρ^{2t} or ρ^{2t+1} disappears from the right-hand side of the equations, and we find

$$\Sigma_{-\infty}^{\infty}\Sigma_{-\infty}^{\infty}\frac{1}{(s_2K'+r_2iK)^{2t}} = \frac{F_t(k^2,-k'^2)}{(2t-1)!},*$$

$$\Sigma_{-\infty}^{\infty}\Sigma_{-\infty}^{\infty}(-)^r\frac{1}{(s_2K'+r_2iK)^{2t}} = (-)^{t-1}\frac{\chi_t(1,k^2)}{(2t-1)!},$$

$$-\Sigma_{-\infty}^{\infty}\Sigma_{-\infty}^{\infty}(-)^s\frac{1}{(s_2K'+r_2iK)^{2t}} = \frac{\chi_t(1,k'^2)}{(2t-1)!},$$

$$-\Sigma_{-\infty}^{\infty}\Sigma_{-\infty}^{\infty}(-)^{r+s}\frac{1}{(s_2K'+r_2iK)^{2t}} = \frac{\chi_t(k^2,-k'^2)}{(2t-1)!};$$

&c. &c.

Corresponding to § 62 we have

$$\Sigma_{-\infty}^{\infty}\Sigma_{-\infty}^{\infty}\frac{1}{(s_2K'+r_2iK)^2} = -\tfrac{1}{8}(\mathrm{i}+\mathrm{g}+\mathrm{e}), \quad (u>v) \qquad = \tfrac{1}{8}(\mathrm{i}'+\mathrm{g}'+\mathrm{e}'), \quad (v>u)$$

$$\Sigma_{-\infty}^{\infty}\Sigma_{-\infty}^{\infty}\frac{1}{(s_2K'+r_1iK)^2} = -\mathrm{e}, \qquad = \mathrm{i}',$$

$$\Sigma_{-\infty}^{\infty}\Sigma_{-\infty}^{\infty}\frac{1}{(s_1K'+r_2iK)^2} = -\mathrm{i}, \qquad = \mathrm{e}',$$

$$\Sigma_{-\infty}^{\infty}\Sigma_{-\infty}^{\infty}\frac{1}{(s_1K'+r_1iK)^2} = -\mathrm{g}, \qquad = \mathrm{g}';$$

and to § 66,

$$i\Sigma_{-\infty}^{\infty}\Sigma_{-\infty}^{\infty}\frac{(-1)^{\frac{1}{2}(r_1-1)}}{s_2K'+r_1iK} = 1, \quad = 1+\frac{(-1)^{\frac{1}{2}(u_1-1)}}{\rho'},$$

$$i\Sigma_{-\infty}^{\infty}\Sigma_{-\infty}^{\infty}\frac{(-1)^{\frac{1}{2}(s_2+r_1-1)}}{s_2K'+r_1iK} = k', \quad = k',$$

&c. &c.

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* The values of the four double summations in which all the terms are positive form the subject of Mr. Forsyth's paper, "Some doubly infinite converging series," *Quarterly Journal*, vol. XXI., pp. 261–280.

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Values of $\Sigma_1^\infty \sigma_{2t-1}(n)\, q^{2n}$, &c., §§ 68–70.

§ 68. We may deduce immediately from § 27 the following system of formulæ, in which, for brevity, x_t is used to denote $\frac{x^t}{t\lfloor}$:

$$\rho\, \mathrm{ns}\rho x - \mathrm{cec}\, x = \Sigma_1^\infty \{\chi_t(1, k^2)\, \rho^{2t} - \chi_t(1, 0)\}\, x_{2t-1},$$

$$\rho\, \mathrm{ds}\rho x - \mathrm{cec}\, x = \Sigma_1^\infty \{\chi_t(k'^2, -k^2)\, \rho^{2t} - \chi_t(1, 0)\}\, x_{2t-1},$$

$$\rho\, \mathrm{cs}\rho x - \cot x = \Sigma_1^\infty (-)^t \{\chi_t(1, k'^2)\, \rho^{2t} - \chi_t(1, 1)\}\, x_{2t-1},$$

$$\rho\, \mathrm{zs}\rho x - \cot x = \{\tfrac{1}{3} - \tfrac{1}{3}(R_i + R_g + R_e)\}\, x - \Sigma_1^\infty \{F_{t+1}(1, k^2)\, \rho^{2t+2} - F_{t+1}(1, 0)\}\, x_{2t+1};$$

$$\rho\, \mathrm{dc}\rho x - \sec x = \rho - 1 + \Sigma_1^\infty \{k'^2 \psi_{t-1}(k'^2, 1)\, \rho^{2t+1} - \psi_{t-1}(1, 1)\}\, x_{2t},$$

$$k'\rho\, \mathrm{nc}\rho x - \sec x = k'\rho - 1 + \Sigma_1^\infty \{k'\, \psi_{t-1}(1, k'^2)\, \rho^{2t+1} - \psi_{t-1}(1, 1)\}\, x_{2t},$$

$$k'\rho\, \mathrm{sc}\rho x - \tan x = \Sigma_1^\infty \{k'\, \phi_{t-1}(1, k'^2)\, \rho^{2t} - \phi_{t-1}(1, 1)\}\, x_{2t-1},$$

$$\rho\, \mathrm{zc}\rho x + \tan x = (1 - R_e)\, x - \Sigma_1^\infty \{k'^2 f_{t-1}(1, k'^2)\, \rho^{2t+2} - f_{t-1}(1, 1)\}\, x_{2t+1};$$

$$k\rho\, \mathrm{cd}\rho x = k\rho - \Sigma_1^\infty (-)^{t-1} k k'^2 \psi_{t-1}(k'^2, -k^2)\, \rho^{2t+1} x_{2t},$$

$$kk'\rho\, \mathrm{sd}\rho x = \Sigma_1^\infty k k' \phi_{t-1}(k^2, -k'^2)\, \rho^{2t} x_{2t-1},$$

$$k'\rho\, \mathrm{nd}\rho x = k'\rho + \Sigma_1^\infty k' k^2 \psi_{t-1}(k^2, -k'^2)\, \rho^{2t+1} x_{2t},$$

$$\rho\, \mathrm{zd}\rho x = -R_g x + \Sigma_1^\infty k^2 k'^2 f_{t-1}(k^2, -k'^2)\, \rho^{2t+2} x_{2t+1};$$

$$k\rho\, \mathrm{sn}\rho x = \Sigma_1^\infty (-)^{t-1} k \phi_{t-1}(1, k^2)\, \rho^{2t} x_{2t-1},$$

$$k\rho\, \mathrm{cn}\rho x = k\rho - \Sigma_1^\infty (-)^{t-1} k \psi_{t-1}(1, k^2)\, \rho^{2t+1} x_{2t},$$

$$\rho\, \mathrm{dn}\rho x = \rho - \Sigma_1^\infty (-)^{t-1} k^2 \psi_{t-1}(k^2, 1)\, \rho^{2t+1} x_{2t},$$

$$\rho\, \mathrm{zn}\rho x = -R_i x - \Sigma_1^\infty (-)^{t-1} k^2 f_{t-1}(1, k^2)\, \rho^{2t+2} x_{2t+1},$$

where R_i, R_g, R_e denote $\frac{4KI}{\pi^2}$, $\frac{4KG}{\pi^2}$, $\frac{4KE}{\pi^2}$.

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§ 69. By equating the coefficients of x^{2t} or x^{2t-1} to those in the doubly infinite series in § 52, we obtain the following summations:

$$2^{2t+1}\Sigma_1^\infty \sigma_{2t-1}(n)\,q^{2n} = F_t(1,k'^2)\,\rho^{2t} - F_t(1,1),$$
$$2^{2t+1}\Sigma_1^\infty \zeta_{2t-1}(n)\,q^{2n} = (-)^{t-1}\{k'^2 f_{t-2}(1,k'^2)\,\rho^{2t} - f_{t-2}(1,1)\},$$
$$2^{2t+1}\Sigma_1^\infty \zeta'_{2t-1}(n)\,q^{2n} = \chi_t(1,k'^2)\,\rho^{2t} - \chi_t(1,1),$$
$$2^{2t+1}\Sigma_1^\infty \xi_{2t-1}(n)\,q^{2n} = (-)^t\{k'\phi_{t-1}(1,k'^2)\,\rho^{2t} - \phi_{t-1}(1,1)\};$$

$$4\,\Sigma_1^\infty \Delta^{2t-1}(n)\,q^n = (-)^{t-1}\{\chi_t(1,k^2)\,\rho^{2t} - \chi_t(1,0)\},$$
$$4\,\Sigma_1^\infty (-)^{n-1}\Delta_{2t-1}(n)\,q^n = (-)^t\{\chi_t(k'^2,-k^2)\,\rho^{2t} - \chi_t(1,0)\},$$
$$4\,\Sigma_1^\infty E_{2t}(n)\,q^n = (-)^t\{k'^2\psi_{t-1}(k'^2,1)\,\rho^{2t+1} - \psi_{t-1}(1,1)\},$$
$$4\,\Sigma_1^\infty (-)^{n-1}E_{2t}(n)\,q^n = (-)^{t-1}\{k'\psi_{t-1}(1,k'^2)\,\rho^{2t+1} - \psi_{t-1}(1,1)\};$$

$$2^{2t+1}\Sigma_1^\infty \Delta'_{2t-1}(n)\,q^n = k^2 f_{t-2}(1,k^2)\,\rho^{2t},$$
$$2^{2t+1}\Sigma_1^\infty (-)^{n-1}\Delta'_{2t-1}(n)\,q^n = k^2k'^2 f_{t-2}(k'^2,-k^2)\,\rho^{2t},$$
$$2^{2t+2}\Sigma_1^\infty E'_{2t}(n)\,q^n = k^2\psi_{t-1}(k^2,1)\,\rho^{2t+1},$$
$$2^{2t+2}\Sigma_1^\infty (-)^{n-1}E'_{2t}(n)\,q^n = (-)^{t-1}k^2k'\psi_{t-1}(k^2,-k'^2)\,\rho^{2t+1};$$

$$4\,\Sigma_1^\infty \Delta_{2t-1}(m)\,q^{\frac{1}{2}m} = k\phi_{t-1}(1,k^2)\,\rho^{2t},$$
$$4\,\Sigma_1^\infty (-)^M\Delta_{2t-1}(m)\,q^{\frac{1}{2}m} = kk'\phi_{t-1}(k'^2,-k^2)\,\rho^{2t},$$
$$4\,\Sigma_1^\infty E_{2t}(m)\,q^{\frac{1}{2}m} = kk'^2\psi_{t-1}(k'^2,-k^2)\,\rho^{2t+1},$$
$$4\,\Sigma_1^\infty (-)^M E_{2t}(m)\,q^{\frac{1}{2}m} = k\phi_{t-1}(1,k^2)\,\rho^{2t+1}.$$

As usual, n represents any number and m any uneven number, and M denotes $\frac{1}{2}(m-1)$. The values of the functions F, f, χ, &c., for the first five or six values of t were given on p. 24.

§ 70. All of these formulæ hold good when t is greater than unity, and, except in the case of the four formulæ derived from the Zeta functions (*i.e.* those involving the functional signs F and f), we may assign to t the value unity. The values of the four excepted series for $t=1$ are readily obtained by equating the coefficients of x in §§ 52 and 67.

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We also obtain, by equating the terms independent of x, the values corresponding to $t=0$ of the six series in which the coefficients depend upon E_{2s} and E'_{2s}.

The values of the ten series in which the suffixes are uneven for the suffix-values 1, 3, 5 are written at length below.

$$2^3\Sigma_1^\infty \sigma(n)\, q^{2n} = \tfrac{1}{3} - \tfrac{1}{3}(R_i + R_\xi + R_\sigma),$$

$$2^3\Sigma_1^\infty \zeta(n)\, q^{2n} = -1 + R_\sigma,$$

$$2^3\Sigma_1^\infty \zeta'(n)\, q^{2n} = -\tfrac{1}{3} + \tfrac{1}{6}(1+k'^2)\rho^2,$$

$$2^3\Sigma_1^\infty \xi(n)\, q^{2n} = 1 - k'\rho^2;$$

$$4\,\Sigma_1^\infty \Delta(n)\, q^n = -\tfrac{1}{6} + \tfrac{1}{6}(1+k^2)\rho^2,$$

$$4\,\Sigma_1^\infty (-)^{n-1}\Delta(n)\, q^n = \tfrac{1}{6} - \tfrac{1}{6}(k'^2 - k^2)\rho^2;$$

$$2^2\Sigma_1^\infty \Delta'(n)\, q^n = -R_i,$$

$$2^2\Sigma_1^\infty (-)^{n-1}\Delta'(n)\, q^n = R_\xi;$$

$$4\,\Sigma_1^\infty \Delta(m)\, q^{\frac{1}{2}m} = k\rho^2,$$

$$4\,\Sigma_1^\infty (-)^M \Delta(m)\, q^{\frac{1}{2}m} = kk'^2\rho^2;$$

$$2^5\Sigma_1^\infty \sigma_3(n)\, q^{2n} = -\tfrac{2}{15} + \tfrac{2}{15}(1 - k^2k'^2)\rho^4,$$

$$2^5\Sigma_1^\infty \zeta_3(n)\, q^{2n} = 2 - 2k'^2\rho^4,$$

$$3^5\Sigma_1^\infty \zeta_3'(n)\, q^{2n} = \tfrac{2}{15} + \tfrac{1}{60}(7 - 22k'^2 + 7k'^4)\rho^4,$$

$$2^5\Sigma_1^\infty \xi_3(n)\, q^{2n} = -2 + k'(1+k'^2)\rho^4;$$

$$4\,\Sigma_1^\infty \Delta_3(n)\, q^n = \tfrac{7}{60} - \tfrac{1}{60}(7 - 22k^2 + 7k^4)\rho^4,$$

$$4\,\Sigma_1^\infty (-)^{n-1}\Delta_3(n)\, q^n = -\tfrac{7}{60} + \tfrac{1}{60}(7k'^4 + 22k^2k'^2 + 7k^4)\rho^4;$$

$$2^5\Sigma_1^\infty \Delta'_3(n)\, q^n = 2k^2\rho^4,$$

$$2^5\Sigma_1^\infty (-)^{n-1}\Delta'_3(n)\, q^n = 2k^2k'^2\rho^4;$$

$$4\,\Sigma_1^\infty \Delta_3(m)\, q^{\frac{1}{2}m} = k(1+k^2)\rho^4,$$

$$4\,\Sigma_1^\infty (-)^M \Delta_3(m)\, q^{\frac{1}{2}m} = kk'(k'^2 - k^2)\rho^4;$$

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$$2^7\Sigma_1^\infty \sigma_5(n)\,q^{2n} = \tfrac{16}{63} - \tfrac{8}{63}(1+k^2)(1+k'^2)(k'^2-k^2)\rho^6,$$
$$2^7\Sigma_1^\infty \zeta_5(n)\,q^{2n} = -2^4 + 2^3k'^2(1+k'^2)\rho^6,$$
$$2^7\Sigma_1^\infty \zeta_5'(n)\,q^{2n} = -\tfrac{16}{63} + \tfrac{1}{126}(31-15k'^2-15k'^4+31k'^6)\rho^6,$$
$$2^7\Sigma_1^\infty \xi_5(n)\,q^{2n} = 16 - k'(1+14k'^2+k'^4)\rho^6;$$

$$4\Sigma_1^\infty \Delta_5(n)\,q^n = -\tfrac{31}{126} + \tfrac{1}{126}(31-15k^2-15k^4+31k^6)\rho^6,$$
$$4\Sigma_1^\infty (-)^{n-1}\Delta_5(n)\,q^n = \tfrac{31}{126} - \tfrac{1}{126}(31k'^6+15k^2k'^4-15k^4k'^2-31k^6)\rho^6;$$

$$2^7\Sigma_1^\infty \Delta'_5(n)\,q^n = 2^3k^2(1+k^2)\rho^6,$$
$$2^7\Sigma_1^\infty (-)^{n-1}\Delta'_5(n)\,q^n = 2^3k^2k'^2(k'^2-k^2)\rho^6;$$

$$4\Sigma_1^\infty \Delta_5(m)\,q^{\frac{1}{2}m} = k(1+14k^2+k^4)\rho^6,$$
$$4\Sigma_1^\infty (-)^M\Delta_5(m)\,q^{\frac{1}{2}m} = kk'(k'^4-14k^2k'^2+k^4)\rho^6.$$

§ 71. The values of the six series in which the suffixes are even are for the suffix zero:

$$4\Sigma_1^\infty E_0(n)\,q^n = 4\Sigma_1^\infty E_0'(n)\,q^n = -1+\rho,$$
$$4\Sigma_1^\infty (-)^{n-1}E_0(n)\,q^n = 4\Sigma_1^\infty (-)^{n-1}E_0'(n)\,q^n = 1-k'\rho,$$
$$4\Sigma_1^\infty E_0(m)\,q^{\frac{1}{2}m} = 4\Sigma_1^\infty (-)^M E_0(m)\,q^{\frac{1}{2}m} = k\rho.$$

Since $E_0(4n+3)=0$ (§ 37) this last formula may be written

$$4\Sigma_0^\infty E_0(4n+1)\,q^{\frac{1}{2}(4n+1)} = k\rho.$$

For the suffixes 2, 4 the values of the series are:

$$4\Sigma_1^\infty E_2(n)\,q^n = 1-k'^2\rho^3,$$
$$4\Sigma_1^\infty (-)^{n-1}E_2(n)\,q^n = -1+k'\rho^3;$$

$$2^4\Sigma_1^\infty E_2'(n)\,q^n = k^2\rho^3,$$
$$2^4\Sigma_1^\infty (-)^{n-1}E_2'(n)\,q^n = k'k^2\rho^3;$$

$$4\Sigma_1^\infty E_2(m)\,q^{\frac{1}{2}m} = kk'^2\rho^3,$$
$$4\Sigma_1^\infty (-)^M E_2(m)\,q^{\frac{1}{2}m} = k\rho^3;$$

$$4\Sigma_1^\infty E_4(n)\,q^n = -5+k'^2(k'^2+4)\rho^5,$$
$$4\Sigma_1^\infty (-)^{n-1}E_4(n)\,q^n = 5-k'(1+4k'^2)\rho^5;$$

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$$2^6\Sigma_1^\infty E_4'(n)\,q^n = k^2(k^2+4)\rho^5,$$
$$2^6\Sigma_1^\infty(-)^{n-1}E_4'(n)\,q^n = kk'^2(4k'^2-k^2)\rho^5;$$
$$4\,\Sigma_1^\infty E_4(m)\,q^{\frac{1}{2}m} = kk'^2(k'^2-4k^2)\rho^5,$$
$$4\,\Sigma_1^\infty(-)^M E_4(m)\,q^{\frac{1}{2}m} = k(1+4k^2)\rho^5.$$

By means of the values of the functions ϕ, ψ, &c., given on p. 24, we may write down immediately as above the values of the sixteen series for higher suffixes, viz., as far as the series involving $\sigma_9(n)$, $\zeta_{13}(n)$, $\zeta_9'(n)$, $\xi_{11}(n)$, $\Delta_9(n)$, $\Delta'_{13}(n)$, $\Delta_{11}(m)$, $E_{13}(n)$, $E'_{13}(n)$, $E_{13}(m)$ respectively.

System of collected results, §§ 72–77.

§ 72. It was shown in § 59 that

$$(2t-1)!\,2^{2t}\Sigma_{-\infty}^{\infty}\Sigma_{-\infty}^{\infty}\frac{1}{(s_2\mu+r_2 i\pi)^{2t}} = F_t(1,k^2)\rho^{2t},$$

and in § 69 that

$$2^{2t+1}\Sigma_1^\infty\sigma_{2t-1}(n)\,q^{2m} = F_t(1,k'^2)\rho^{2t} - F_t(1,1),$$

where
$$F_t(1,1) = (-)^t\frac{2^{2t-1}B_t}{t} \quad (\S 31).$$

By § 6 we have also

$$\Sigma_1^\infty\sigma_{2t-1}(n)\,q^{2n} = \Sigma_1^\infty\frac{n^{2t-1}q^{2n}}{1-q^{2n}}.$$

We have thus obtained the equalities

$$(2t-1)!\,2^{2t}\Sigma_{-\infty}^{\infty}\Sigma_{-\infty}^{\infty}\frac{1}{(s_2\mu+r_2 i\pi)^{2t}} = (-)^t 2^{2t-1}\frac{B_t}{t} + 2^{2t+1}\Sigma_1^\infty\sigma_{2t-1}(n)\,q^{2n}$$
$$= (-)^t 2^{2t-1}\frac{B_t}{t} + 4\Sigma_1^\infty\frac{(2n)^{2t-1}q^{2n}}{1-q^{2n}} = F_t(1,k'^2)\rho^{2t}.$$

These equations may also be written (§ 64) in the slightly more compendious form

$$(2t-1)!\,2^{2t-1}\Sigma_2^\infty\Sigma_{-\infty}^{\infty}\frac{1}{(s_2\mu+r_2 i\pi)^{2t}} = 2^{2t-1}\Sigma_1^\infty\sigma_{2t-1}(n)\,q^{2n}$$
$$= \Sigma_1^\infty\frac{(2n)^{2t-1}q^{2n}}{1-q^{2n}} = \tfrac{1}{4}\{F_t(1,k'^2)\rho^{2t} - F_t(1,1)\}.$$

§ 73. Adopting this second form, we readily derive from §§ 64 and 69 the following system of equalities.

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$$(2t-1)!\,2^{2t-1}\sum_2^\infty\sum_{-\infty}^\infty\frac{1}{(s_2\mu+r_2 i\pi)^{2t}}=\sum_1^\infty\frac{(2n)^{2t-1}q^{2n}}{1-q^{2n}}=2^{2t-1}\sum_1^\infty\sigma_{2t-1}(n)\,q^{2n}=\tfrac{1}{4}\{F_t(1,k'^2)\,\rho^{2t}-F_t(1,1)\},$$

$$-(2t-1)!\,2^{2t-1}\sum_2^\infty\sum_{-\infty}^\infty\frac{(-1)^{\frac{1}{2}s_2}}{(s_2\mu+r_2 i\pi)^{2t}}=\sum_1^\infty\frac{(2n)^{2t-1}q^{2n}}{1+q^{2n}}=2^{2t-1}\sum_1^\infty\zeta'_{2t-1}(n)\,q^{2n}=\tfrac{1}{4}\{\chi_t(1,k'^2)\,\rho^{2t}-\chi_t(1,1)\},$$

$$-(2t-1)!\,2^{2t-1}\sum_2^\infty\sum_{-\infty}^\infty\frac{1}{(s_2\mu+r_1 i\pi)^{2t}}=\sum_1^\infty(-)^{n-1}\frac{(2n)^{2t-1}q^{2n}}{1-q^{2n}}=2^{2t-1}\sum_1^\infty\zeta_{2t-1}(n)\,q^{2n}=(-)^{t-1}\tfrac{1}{4}\{k'^2f_{t-2}(1,k'^2)\,\rho^{2t}-f_{t-2}(1,1)\},$$

$$(2t-1)!\,2^{2t-1}\sum_2^\infty\sum_{-\infty}^\infty\frac{(-1)^{\frac{1}{2}s_2}}{(s_2\mu+r_1 i\pi)^{2t}}=\sum_1^\infty(-)^{n-1}\frac{(2n)^{2t-1}q^{2n}}{1+q^{2n}}=2^{2t-1}\sum_1^\infty\xi_{2t-1}(n)\,q^{2n}=(-)^t\tfrac{1}{4}\{k'\phi_{t-1}(1,k'^2)\rho^{2t}-\phi_{t-1}(1,1)\};$$

$$(2t-1)!\,2^{2t-1}\sum_1^\infty\sum_{-\infty}^\infty\frac{1}{(s_1\mu+r_2 i\pi)^{2t}}=\sum_1^\infty\frac{(2n)^{2t-1}q^{n}}{1-q^{2n}}=2^{2t-1}\sum_1^\infty\Delta'_{2t-1}(n)\,q^{n}=\tfrac{1}{4}k^2f_{t-2}(1,k^2)\,\rho^{2t},$$

$$-(2t-1)!\,2^{2t-1}\sum_1^\infty\sum_{-\infty}^\infty\frac{1}{(s_1\mu+r_1 i\pi)^{2t}}=\sum_1^\infty(-)^{n-1}\frac{(2n)^{2t-1}q^{n}}{1-q^{2n}}=2^{2t-1}\sum_1^\infty(-)^{n-1}\Delta'_{2t-1}(n)\,q^{n}=\tfrac{1}{4}k^2k'^2f_{t-2}(k'^2,-k^2)\,\rho^{2t};$$

$$(2t-1)!\,2^{2t-1}\sum_2^\infty\sum_{-\infty}^\infty\frac{(-1)^{\frac{1}{2}r_2}}{(s_2\mu+r_2 i\pi)^{2t}}=\sum_1^\infty\frac{m^{2t-1}q^{m}}{1-q^{m}}=\sum_1^\infty\Delta_{2t-1}(n)\,q^{n}=(-)^{t-1}\tfrac{1}{4}\{\chi_t(1,k^2)\,\rho^{2t}-\chi_t(1,0)\},$$

$$-(2t-1)!\,2^{2t-1}\sum_2^\infty\sum_{-\infty}^\infty\frac{(-1)^{\frac{1}{2}(r_2+s_2)}}{(s_2\mu+r_2 i\pi)^{2t}}=\sum_1^\infty\frac{m^{2t-1}q^{m}}{1+q^{m}}=\sum_1^\infty(-)^{n-1}\Delta_{2t-1}(n)\,q^{n}=(-)^t\tfrac{1}{4}\{\chi_t(k'^2,-k^2)\,\rho^{2t}-\chi_t(1,0)\};$$

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$$(2t-1)!\,2^{2t-1}\,\Sigma_1^\infty\Sigma_{-\infty}^\infty\frac{(-1)^{\frac{1}{2}r_2}}{(s_1\mu+r_2i\pi)^{2t}}=\Sigma_1^\infty\frac{m^{2t-1}q^{\frac{1}{2}m}}{1-q^m}=\Sigma_1^\infty\Delta_{2t-1}(m)\,q^{\frac{1}{2}m}=\tfrac{1}{4}k\phi_{t-1}(1,k^2)\,\rho^{2t},$$

$$(2t-1)!\,2^{2t-1}\,\Sigma_1^\infty\Sigma_{-\infty}^\infty\frac{(-1)^{\frac{1}{2}(r_1+s_1)}i}{(s_1\mu+r_1i\pi)^{2t}}=\Sigma_1^\infty(-)^M\frac{m^{2t-1}q^{\frac{1}{2}m}}{1+q^m}=\Sigma_1^\infty(-)^M\Delta_{2t-1}(m)q^{\frac{1}{2}m}=\tfrac{1}{4}kk'\phi_{t-1}(k'^2,-k^2)\,\rho^{2t};$$

$$(2t)!\;\;2^{2t}\;\;\Sigma_1^\infty\Sigma_{-\infty}^\infty\frac{(-1)^{\frac{1}{2}(s_1-1)}}{(s_1\mu+r_2i\pi)^{2t+1}}=\Sigma_1^\infty\frac{(2n)^{2t}q^n}{1+q^{2n}}=2^{2t}\,\Sigma_1^\infty E'_{2t}(n)\,q^n=\tfrac{1}{4}k^3\psi_{t-1}(k^2,1)\,\rho^{2t+1},$$

$$-\;(2t)!\;\;2^{2t}\;\;\Sigma_1^\infty\Sigma_{-\infty}^\infty\frac{(-1)^{\frac{1}{2}(s_1-1)}}{(s_1\mu+r_1i\pi)^{2t+1}}=\Sigma_1^\infty(-)^{n-1}\frac{(2n)^{2t}q^n}{1+q^{2n}}=2^{2t}\,\Sigma_1^\infty(-)^{n-1}E'_{2t}(n)\,q^n=(-)^{t-1}\tfrac{1}{4}k^3k'\psi_{t-1}(k^2,-k'^2)\,\rho^{2t+1};$$

$$(2t)!\;\;2^{2t}\;\;\Sigma_2^\infty\Sigma_{-\infty}^\infty\frac{(-1)^{\frac{1}{2}(r_1-1)}i}{(s_2\mu+r_1i\pi)^{2t+1}}=\Sigma_1^\infty(-)^M\frac{m^{2t}q^m}{1-q^m}=\Sigma_1^\infty E_{2t}(n)\,q^n=(-)^t\;\tfrac{1}{4}\{k'^2\psi_{t-1}(k'^2,1)\rho^{2t+1}-\psi_{t-1}(1,1)\},$$

$$-\;(2t)!\;\;2^{2t}\;\;\Sigma_2^\infty\Sigma_{-\infty}^\infty\frac{(-1)^{\frac{1}{2}(s_2+r_1-1)}i}{(s_2\mu+r_1i\pi)^{2t+1}}=\Sigma_1^\infty(-)^M\frac{m^{2t}q^m}{1+q^m}=\Sigma_1^\infty(-)^{n-1}E_{2t}(n)\,q^n=(-)^{t-1}\tfrac{1}{4}\{k'\psi_{t-1}(1,k'^2)\rho^{2t+1}-\psi_{t-1}(1,1)\};$$

$$(2t)!\;\;2^{2t}\;\;\Sigma_1^\infty\Sigma_{-\infty}^\infty\frac{(-1)^{\frac{1}{2}(r_1-1)}i}{(s_1\mu+r_1i\pi)^{2t+1}}=\Sigma_1^\infty(-)^M\frac{m^{2t}q^{\frac{1}{2}m}}{1-q^m}=\Sigma_1^\infty E_{2t}(m)\,q^{\frac{1}{2}m}=\tfrac{1}{4}kk'^2\psi_{t-1}(k'^2,-k^2)\,\rho^{2t+1},$$

$$(2t)!\;\;2^{2t}\;\;\Sigma_1^\infty\Sigma_{-\infty}^\infty\frac{(-1)^{\frac{1}{2}(r_2+s_1-1)}}{(s_1\mu+r_2i\pi)^{2t+1}}=\Sigma_1^\infty\frac{m^{2t}q^{\frac{1}{2}m}}{1+q^m}=\Sigma_1^\infty(-)^ME_{2t}(m)q^{\frac{1}{2}m}=\tfrac{1}{4}k\psi_{t-1}(1,k^2)\,\rho^{2t+1}.$$

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§ 74. We may deduce at sight the corresponding expressions for the double summations when both sets of limits are $\pm\infty$ (*i. e.*, as in the first form in § 72) by simply multiplying throughout by 4, and transposing the second term of the last column, in the eight cases where there is such a term, to the other columns. We thus convert $(2t-1)!\,2^{2t-1}\Sigma_2^\infty\Sigma_{-\infty}^\infty$ into $(2t-1)!\,2^{2t}\Sigma_{-\infty}^\infty\Sigma_{-\infty}^\infty$; &c. For example, taking the first equation, we have

$$(2t-1)!\,2^{2t}\Sigma_{-\infty}^\infty\Sigma_{-\infty}^\infty\frac{1}{(s_2\mu+r_2i\pi)^{2t}}=F_t(1,1)+4\Sigma_1^\infty\frac{(2n)^{2t-1}q^{2n}}{1-q^{4n}}$$
$$=F_t(1,1)+2^{2t+1}\Sigma_1^\infty\sigma_{2t-1}(n)\,q^{2n}=F_t(1,k'^2)\,\rho^{2t}.$$

The values of the quantities $F(1,1)$, &c., which then occur as constant terms in eight of the q-series, are as follows (§ 31):

$$F_t(1,1)=(-)^t\,2^{2t-1}\frac{B_t}{t},\quad(-)^{t-1}f_{t-2}(1,1)=(-)^{t-1}2^{2t-1}(2^{2t}-1)\frac{B_t}{t},$$

$$\chi_t(1,1)=(-)^{t-1}2^{2t-1}\frac{B_t}{t},\quad(-)^t\,\phi_{t-1}(1,1)=(-)^t\,2^{2t-1}(2^{2t}-1)\frac{B_t}{t},$$

$$(-)^{t-1}\chi_t(1,0)=(-)^{t-1}(2^{2t-1}-1)\frac{B_t}{t},\quad(-)^t\psi_{t-1}(1,1)=(-)^tE_t,$$

where B_t and E_t denote respectively the t^{th} Bernoullian and Eulerian numbers.

§ 75. Four of the equalities in § 73 do not hold good when $t=1$ (§ 61). Corresponding to these special cases we have from §§ 65 and 70, if the r-limits be infinitely larger than the s-limits,

$$\Sigma_2^\infty\Sigma_{-\infty}^\infty\frac{1}{(s_2\mu+r_2i\pi)^2}=\Sigma_1^\infty\frac{nq^{2n}}{1-q^{4n}}=\Sigma_1^\infty\sigma(n)\,q^{2n}$$
$$=\tfrac{1}{24}-\tfrac{1}{24}(R_i+R_g+R_d),$$

$$-\Sigma_2^\infty\Sigma_{-\infty}^\infty\frac{1}{(s_2\mu+r_1i\pi)^2}=\Sigma_1^\infty(-)^{n-1}\frac{nq^{2n}}{1-q^{2n}}=\Sigma_1^\infty\zeta(n)\,q^{2n}$$
$$=-\tfrac{1}{8}+\tfrac{1}{8}R_d,$$

$$\Sigma_1^\infty\Sigma_{-\infty}^\infty\frac{1}{(s_1\mu+r_2i\pi)^2}=\Sigma_1^\infty\frac{nq^n}{1-q^{2n}}=\Sigma_1^\infty\Delta'(n)\,q^n$$
$$=-\tfrac{1}{8}R_i,$$

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$$-\Sigma_1^\infty \Sigma_{-\infty}^\infty \frac{1}{(s_1\mu + r_1 i\pi)^2} = \Sigma_1^\infty (-)^{n-1} \frac{nq^n}{1-q^{2n}} = \Sigma_1^\infty (-)^{n-1} \Delta'(n)\, q^n$$
$$= \tfrac{1}{8} R_q.$$

To deduce the expressions for the double summations when both limits are $\pm\infty$ we multiply throughout 2, and transpose the terms $\frac{1}{12}$ and $-\frac{1}{4}$, which occur in the first two equations, to the other columns. For example,

$$\Sigma_{-\infty}^\infty \Sigma_{-\infty}^\infty \frac{1}{(s_2\mu + r_2 i\pi)^2} = -\tfrac{1}{12} + \Sigma_1^\infty \frac{2nq^{2n}}{1-q^{2n}} = -\tfrac{1}{12} + 2\Sigma_1^\infty \sigma(n) q^{2n}$$
$$= -\tfrac{1}{12}(R_i + R_q + R_e).$$

§ 76. Corresponding to the case $t=0$ of the last six equalities in § 73, we have, from §§ 66 and 71,

$$\Sigma_2^\infty \Sigma_{-\infty}^\infty \frac{(-1)^{\frac{1}{2}(r_1-1)}\, i}{s_2\mu + r_1 i\pi} = \Sigma_1^\infty \Sigma_{-\infty}^\infty \frac{(-1)^{\frac{1}{2}(s_1-1)}}{s_1\mu + r_2 i\pi}$$
$$= \Sigma_1^\infty (-)^M \frac{q^m}{1-q^{2m}} = \Sigma_1^\infty \frac{q^n}{1+q^{2n}}$$
$$= \Sigma_1^\infty E_0(n)\, q^n = \Sigma_1^\infty E_0'(n)\, q^n = -\tfrac{1}{4} + \tfrac{1}{4}\rho,$$

$$\Sigma_2^\infty \Sigma_{-\infty}^\infty \frac{(-1)^{\frac{1}{2}(s_2+r_1-1)}\, i}{s_2\mu + r_1 i\pi} = \Sigma_1^\infty \Sigma_{-\infty}^\infty \frac{(-1)^{\frac{1}{2}(s_1-1)}}{s_1\mu + r_1 i\pi}$$
$$= -\Sigma_1^\infty (-)^M \frac{q^m}{1+q^{2m}} = \Sigma_1^\infty (-)^n \frac{q^n}{1+q^{2n}}$$
$$= \Sigma_1^\infty (-)^n E_0(n)\, q^n = \Sigma_1^\infty (-)^n E_0'(n)\, q^n = -\tfrac{1}{4} + \tfrac{1}{4} k'\rho,$$

$$\Sigma_1^\infty \Sigma_{-\infty}^\infty \frac{(-1)^{\frac{1}{2}(r_2+s_1-1)}}{s_1\mu + r_2 i\pi} = \Sigma_1^\infty \Sigma_{-\infty}^\infty \frac{(-1)^{\frac{1}{2}(r_1-1)}\, i}{s_1\mu + r_1 i\pi}$$
$$= \Sigma_1^\infty (-)^M \frac{q^{\frac{1}{2}m}}{1-q^{2m}} = \Sigma_1^\infty \frac{q^{\frac{1}{2}m}}{1+q^{2m}}$$
$$= \Sigma_1^\infty E_0(m)\, q^{\frac{1}{2}m} = \Sigma_1^\infty (-)^M E_0(m)\, q^{\frac{1}{2}m}$$
$$= \Sigma_0^\infty E_0(4n+1)\, q^{\frac{1}{2}(4n+1)} = \tfrac{1}{4} k\rho.$$

In the above formulæ the r-limits are supposed to be infinitely larger than the s-limits. In the two double summations, in which the denominator is $s_2\mu + r_1 i\pi$, the s-limits are supposed to be of the form $\pm(4n+3)$.

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§ 77. In connexion with these formulæ it may be remarked that they may be readily proved directly by effecting the double summations. Thus, taking the first formula, we have

$$2i\,\Sigma_{-\infty}^{\infty}\Sigma_{-\infty}^{\infty}\frac{(-1)^{\frac{1}{2}(r_1-1)}}{s_2\mu+r_1 i\pi}=1+4i\;\Sigma_{2}^{\infty}\Sigma_{-\infty}^{\infty}\frac{(-1)^{\frac{1}{2}(r_1-1)}}{s_2\mu+r_1 i\pi}$$

$$=1+8\pi\,\Sigma_{2}^{\infty}\Sigma_{1}^{\infty}\frac{(-)^{\frac{1}{2}(r_1-1)}\,r_1}{s_2^{2}\mu^{2}+r_1^{2}\pi^{2}}.$$

Effecting the summation first with respect to r_1 (which is equivalent to assuming that the limiting values of r_1 are infinitely larger than the largest value of s_2), we find (§ 44, putting $x=0$) that this expression

$$=1+2\,\Sigma_{2}^{\infty}\operatorname{sech}\tfrac{1}{2}s_2\mu=1+4\,\Sigma_{1}^{\infty}\frac{q^{n}}{1+q^{2n}}=\rho.$$

If we effect the summation first with regard to s_1 (which is equivalent to assuming that the s-limits are infinitely larger than the r-limits) we find (§ 44) that the expression

$$=1+\frac{8}{\mu}\,\Sigma_{1}^{\infty}(-)^{\frac{1}{2}(r_1-1)}\left\{\tfrac{1}{4}\pi\coth\tfrac{1}{2}r_1\mu'-\frac{\mu}{2r_1\pi}\right\},\text{ where }\mu'=\frac{\pi^{2}}{\mu}$$

$$=1-\frac{4}{\pi}\,\Sigma_{1}^{\infty}(-)^{\frac{1}{2}(r_1-1)}\frac{1}{r_1}+\frac{2\pi}{\mu}\,\Sigma_{1}^{\infty}(-)^{\frac{1}{2}(r_1-1)}\coth\tfrac{1}{2}r_1\mu'$$

$$=\frac{2\pi}{\mu}\,\Sigma_{1}^{\infty}(-)^{\frac{1}{2}(r_1-1)}\left\{1+\frac{2q'^{r_1}}{1-q'^{r_1}}\right\}$$

$$=\frac{4\pi}{\mu}\,\Sigma_{1}^{\infty}(-)^{\frac{1}{2}(r_1-1)}\frac{q'^{r_1}}{1-q'^{r_1}}+\frac{\pi}{\mu}\left\{1+(-1)^{\frac{1}{2}(u_1-1)}\right\}$$

$$=\frac{\pi}{\mu}(\rho'-1)+\frac{\pi}{\mu}\left\{1+(-1)^{\frac{1}{2}(u_1-1)}\right\}=\rho+(-)^{\frac{1}{2}(u_1-1)}\frac{\pi}{\mu};$$

which agrees with § 66.

Results obtained by differentiation with respect to μ, §§ 78, 79.

§ 78. By differentiating p times with respect to μ the formulæ of §§ 73–76 we may deduce the values of the double summations of terms of the forms

$$\frac{s_2^{\,p}}{(s_2\mu+r_2 i\pi)^{2i+p}},\qquad\frac{(-)^{\frac{1}{2}s_2}s_2^{\,p}}{(s_2\mu+r_2 i\pi)^{2i+p}},\ \&c.$$

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The operator $-\frac{d}{d\mu}$ is the same as the operator $q\frac{d}{dq}$, which was considered in the *Quarterly Journal*, vol. XX., p. 353. A formula is there given for the result of operating upon any expression of the form $k^m k'^n K^r I^\alpha G^\beta E^\gamma$.* This operator may be conveniently denoted by ϑ.

For example, by operating with ϑ upon the results obtained by putting $t=2$ in the first formula of § 73 and $t=1$ in the second, viz.

$$6\Sigma_{-\infty}^{\infty}\Sigma_{-\infty}^{\infty}\frac{1}{(s_2\mu+r_2 i\pi)^4}=\tfrac{1}{2}\tfrac{1}{4}\tfrac{1}{0}+2\Sigma_1^\infty\sigma_3(n)\,q^{2n}=\tfrac{1}{2}\tfrac{1}{4}\tfrac{1}{0}(1-k^2k'^2)\rho^4,$$

and

$$-\Sigma_{-\infty}^{\infty}\Sigma_{-\infty}^{\infty}\frac{(-1)^{\frac{1}{2}s_2}}{(s_2\mu+r_2 i\pi)^2}=\tfrac{1}{1}\tfrac{1}{2}+2\Sigma_1^\infty\zeta'(n)\,q^{2n}=\tfrac{1}{2}\tfrac{1}{4}(1+k'^2)\rho^2,$$

we find

$$6\Sigma_{-\infty}^{\infty}\Sigma_{-\infty}^{\infty}\frac{s_2}{(s_2\mu+r_2 i\pi)^5}=\Sigma_1^\infty n\sigma_3(n)\,q^{2n}$$
$$=\tfrac{1}{4}\tfrac{1}{8}\tfrac{1}{0}(1-k^2k'^2)\rho^4R_\varrho-\tfrac{1}{9}\tfrac{1}{6}\tfrac{1}{0}k^2k'^2(k'^2-k^2)\rho^6,$$

and

$$-\Sigma_{-\infty}^{\infty}\Sigma_{-\infty}^{\infty}\frac{(-)^{\frac{1}{2}s_2}s_2}{(s_2\mu+r_2 i\pi)^3}=2\Sigma_1^\infty n\zeta'(n)q^{2n}=\tfrac{1}{4}\tfrac{1}{8}(1+k'^2)\rho^2R_\varrho-\tfrac{1}{4}\tfrac{1}{8}k^2k'^2\rho^4.$$

Similarly from the formula (§§ 62 and 70)

$$\Sigma_{-\infty}^{\infty}\Sigma_{-\infty}^{\infty}\frac{1}{(s_2\mu+r_2 i\pi)^2}=-\tfrac{1}{1}\tfrac{1}{2}+2\Sigma_1^\infty\sigma(n)\,q^{2n}=-\tfrac{1}{1}\tfrac{1}{2}(R_i+R_\varrho+R_\epsilon),$$

we find, by operating with ϑ and ϑ^2,

$$\Sigma_{-\infty}^{\infty}\Sigma_{-\infty}^{\infty}\frac{s_2}{(s_2\mu+r_2 i\pi)^3}=2\Sigma_1^\infty n\sigma(n)q^{2n}=-\tfrac{1}{4}\tfrac{1}{8}(R_\varrho R_\epsilon+R_\epsilon R_i+R_i R_\varrho),$$

and

$$3\Sigma_{-\infty}^{\infty}\Sigma_{-\infty}^{\infty}\frac{s_2^2}{(s_2\mu+r_2 i\pi)^4}=4\Sigma_1^\infty n^2\sigma(n)q^{2n}=-\tfrac{1}{1}\tfrac{1}{6}R_iR_\varrho R_\epsilon,$$

the r-limits being supposed infinitely larger than the s-limits in all three formulæ.

---

* This formula may be written:

$$\vartheta\,(k^m k'^n\rho^r R_i^\alpha R_\varrho^\beta R_\epsilon^\gamma)=\tfrac{1}{2}k^m k'^m\rho^r R_i^\alpha R_\varrho^\beta R_\epsilon^\gamma\Big\{(\beta+\gamma+n)\,R_i$$
$$+(\gamma+\alpha-n-m+r)\,R_\varrho+(\alpha+\beta+m)\,R_\epsilon-\alpha\frac{R_\varrho R_\epsilon}{R_i}-\beta\frac{R_\epsilon R_i}{R_\varrho}-\gamma\frac{R_iR_\varrho}{R_\epsilon}\Big\}.$$

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§ 79. We cannot of course write down the general formulæ which result from operating p times upon the expressions involving k and ρ in § 73, but we can always express the values of the double summations as q-series.

For example, we have

$$(2t+p-1)!\,\Sigma_{-\infty}^{\infty}\Sigma_{-\infty}^{\infty}\frac{{s_3}^p}{(s_3\mu+r_2 i\pi)^{2t+p}}=2^{p+1}\Sigma_1^{\infty}n^p\sigma_{2t-1}(n)\,q^{2n},$$

$$-(2t+p-1)!\,\Sigma_{-\infty}^{\infty}\Sigma_{-\infty}^{\infty}\frac{(-)^{\frac{1}{2}s_2}{s_3}^p}{(s_3\mu+r_2 i\pi)^{2t+p}}=2^{p+1}\Sigma_1^{\infty}n^p\zeta'_{2t-1}(n)\,q^{n},$$

&c. &c.

The special case when the modulus is $\frac{1}{\sqrt{2}}$, §§ 80, 81.

§ 80. By putting $\mu=\pi$ in the results given in §§ 73–76 we obtain expressions for the sums of the reciprocals of powers of all the complex numbers.

The case $\mu=\pi$ corresponds to $K=K'$, so that the modulus is $\frac{1}{\sqrt{2}}$. It is convenient therefore, before proceeding to these formulæ, to give some results having reference to this value of the modulus.

Using the exponent zero, to indicate that the modulus is supposed to be $\frac{1}{\sqrt{2}}$, we easily find

$$I^\circ=\frac{\pi}{4K^\circ}-\tfrac{1}{2}K^\circ,\quad G^\circ=\frac{\pi}{4K^\circ},\quad E^\circ=\frac{\pi}{4K^\circ}+\tfrac{1}{2}K^\circ.$$

§ 81. If we denote the length of one of the loops of the lemniscate $r^2=a^2\cos 2\theta$ by $a\omega$*, it can be shown that $\omega=K^\circ\sqrt{2}$.

---

* The quantity ω in the lemniscate corresponds exactly to π in the circle, *i.e.* if we suppose a lemniscate described having its node coincident with the centre of the circle, and its axis $2a$ coincident with the diameter of the circle, the whole length of the lemniscate is $2a\omega$ and that of the circle is $2a\pi$.

In denoting the value of the integral $\int_0^1\frac{dx}{\sqrt{(1-x^4)}}$ by $\frac{1}{2}\omega$, I have followed the notation used by Gauss in his posthumous paper, "*De curvâ lemniscatâ*" (Werke, vol. iii., p. 413).

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Expressing the results in terms of ω, we have

$$K^\circ = \frac{\omega}{\sqrt{2}}, \qquad \rho^\circ = \frac{\omega\sqrt{2}}{\pi},$$

$$R_i^\circ = \frac{1}{\pi} - \frac{\omega^2}{\pi^2}, \qquad R_g^\circ = \frac{1}{\pi}, \qquad R_e^\circ = \frac{1}{\pi} + \frac{\omega^2}{\pi^2},$$

$$I^\circ = \frac{1}{2\sqrt{2}}\left(\frac{\pi}{\omega} - \omega\right), \qquad G^\circ = \frac{1}{2\sqrt{2}}\frac{\pi}{\omega}, \qquad E^\circ = \frac{1}{2\sqrt{2}}\left(\frac{\pi}{\omega} + \omega\right),$$

$$i^\circ = \tfrac{1}{2}\left(\frac{\pi}{\omega^2} - 1\right), \qquad g^\circ = \tfrac{1}{2}\frac{\pi}{\omega^2}, \qquad e^\circ = \tfrac{1}{2}\left(\frac{\pi}{\omega^2} + 1\right).$$

The numerical values of K° and ω to eight places of decimals are

$$K^\circ = 1{\cdot}85407968, \qquad \omega = 2{\cdot}62205756.^*$$

It may be remarked that K° may be expressed in terms of the Gamma Function by means of the formula

$$K^\circ = \tfrac{1}{4}\frac{\Gamma^2(\frac{1}{4})}{\Gamma(\frac{1}{2})} = \tfrac{1}{4}\frac{\Gamma^2(\frac{1}{4})}{\sqrt{\pi}}.$$

Sums of the reciprocals of powers of the complex numbers, §§ 82–86.

§ 82. By putting $k = \frac{1}{\sqrt{2}}$ in the system of results in § 73 we obtain the following equalities which express the sums of the reciprocals of the complex numbers in terms of ω, and also as series proceeding by powers of $e^{-\pi}$. The first group relates to the even numbers, the second to the semi-even numbers, and the third to the uneven numbers.

Even numbers.

$$\Sigma_{-\infty}^{\infty}\Sigma_{-\infty}^{\infty}\frac{1}{(s_2 + ir_2)^{2t}} = \frac{F_t(1,-1)\,\omega^{2t}}{(2t-1)!\,2^{2t}}$$

$$= \frac{\pi^{2t}}{(2t-1)!\,2^{2t}}\{F_t(1,1) + 2^{2t+1}\Sigma_1^{\infty}\sigma_{2t-1}(n)\,e^{-2n\pi}\},$$

* In the memoir cited above Gauss has given the value of $\frac{1}{2}\omega$ to 23 places of decimals, and the values of various powers of $\frac{1}{2}\omega$ to ten or more places.

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$$-\Sigma_{-\infty}^{\infty}\Sigma_{-\infty}^{\infty}\frac{(-1)^{\frac{1}{2}s_2}}{(s_2+ir_2)^{2t}}=(-)^{t-1}\Sigma_{-\infty}^{\infty}\Sigma_{-\infty}^{\infty}\frac{(-1)^{\frac{1}{2}r_2}}{(s_2+ir_2)^{2t}}=\frac{\chi_t(1,2)\,\omega^{2t}}{(2t-1)\,!\,2^{2t}}$$

$$=\frac{\pi^{2t}}{(2t-1)\,!\,2^{2t}}\{\chi_t(1,1)+2^{2t+1}\Sigma_1^{\infty}\zeta'_{2t-1}(n)\,e^{-2n\pi}\}$$

$$=\frac{\pi^{2t}}{(2t-1)\,!\,2^{2t}}\{\chi_t(1,0)+(-)^{t-1}4\,\Sigma_1^{\infty}\Delta_{2t-1}(n)e^{-n\pi}\},$$

$$-\Sigma_{-\infty}^{\infty}\Sigma_{-\infty}^{\infty}\frac{(-1)^{\frac{1}{2}(r_2+s_2)}}{(s_2+ir_2)^{2t}}=\frac{\chi_t(1,-1)\,\omega^{2t}}{(2t-1)\,!\,2^{2t}}$$

$$=\frac{\pi^{2t}}{(2t-1)\,!\,2^{2t}}\{\chi_t(1,0)+(-)^t4\Sigma_1^{\infty}(-)^{n-1}\Delta_{2t-1}(n)\,e^{-n\pi}\}.$$

Semi-even numbers.

$$-\Sigma_{-\infty}^{\infty}\Sigma_{-\infty}^{\infty}\frac{1}{(s_1+ir_1)^{2t}}=\frac{f_{t-2}(1,-1)\,\omega^{2t}}{(2t-1)\,!\,2^{2t}}$$

$$=\frac{2\pi^{2t}}{(2t-1)\,!}\,\Sigma_1^{\infty}(-)^{n-1}\Delta'_{2t-1}(n)\,e^{-n\pi},$$

$$\Sigma_{-\infty}^{\infty}\Sigma_{-\infty}^{\infty}\frac{(-1)^{\frac{1}{2}(r_1-1)}i}{(s_1+ir_1)^{2t+1}}=(-)^t\Sigma_{-\infty}^{\infty}\Sigma_{-\infty}^{\infty}\frac{(-1)^{\frac{1}{2}(s_1-1)}}{(s_1+ir_1)^{2t+1}}=\frac{\psi_{t-1}(1,-1)\omega^{2t+1}}{(2t)\,!\,2^{2t+1}}$$

$$=\frac{\pi^{2t+1}}{(2t)\,!\,2^{2t-1}}\,\Sigma_1^{\infty}E_{2t}(m)\,e^{-\frac{1}{2}m\pi}$$

$$=(-)^{t-1}\frac{2\pi^{2t+1}}{(2t)\,!}\,\Sigma_1^{\infty}(-)^{n-1}E'_{2t}(n)\,e^{-n\pi},$$

$$\Sigma_{-\infty}^{\infty}\Sigma_{-\infty}^{\infty}\frac{(-1)^{\frac{1}{2}(r_1+s_1)}\,i}{(s_1+ir_1)^{2t}}=\frac{\phi_{t-1}(1,-1)\,\omega^{2t}}{(2t-1)\,!\,2^{2t}}$$

$$=\frac{\pi^{2t}}{(2t-1)\,!\,2^{2t-2}}\,\Sigma_1^{\infty}(-)^M\Delta_{2t-1}(m)\,e^{-\frac{1}{2}m\pi}.$$

Uneven numbers.

$$\Sigma_{-\infty}^{\infty}\Sigma_{-\infty}^{\infty}\frac{1}{(s_1+ir_2)^{2t}}=(-)^t\Sigma_{-\infty}^{\infty}\Sigma_{-\infty}^{\infty}\frac{1}{(s_2+ir_1)^{2t}}=\frac{f_{t-2}(1,2)\,\omega^{2t}}{(2t-1)\,!\,2^{2t-1}}$$

$$=\frac{2\pi^{2t}}{(2t-1)\,!}\,\Sigma_1^{\infty}\Delta'_{2t-1}(n)\,e^{-n\pi}$$

$$=\frac{\pi^{2t}}{(2t-1)\,!\,2^{2t}}\{f_{t-2}(1,1)+(-)^{t-1}2^{2t+1}\Sigma_1^{\infty}\zeta_{2t-1}(n)\,e^{-2n\pi}\},$$

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$$\Sigma_{-\infty}^{\infty}\Sigma_{-\infty}^{\infty}\frac{(-1)^{\frac{1}{2}r_2}}{(s_1+ir_2)^{2t}}=(-)^t\Sigma_{-\infty}^{\infty}\Sigma_{-\infty}^{\infty}\frac{(-1)^{\frac{1}{2}s_2}}{(s_2+ir_1)^{2t}}=\frac{\phi_{t-1}(1,2)\,\omega^{2t}}{(2t-1)!\,2^{2t-\frac{1}{2}}}$$

$$=\frac{\pi^{2t}}{(2t-1)!\,2^{2t-2}}\,\Sigma_1^{\infty}\Delta_{2t-1}(m)\,e^{-\frac{1}{2}m\pi}$$

$$=\frac{\pi^{2t}}{(2t-1)!\,2^{2t}}\{\phi_{t-1}(1,1)+(-)^t\,2^{2t+1}\Sigma_1^{\infty}\xi_{2t-1}(n)\,e^{-2n\pi}\},$$

$$\Sigma_{-\infty}^{\infty}\Sigma_{-\infty}^{\infty}\frac{(-1)^{\frac{1}{2}(s_1-1)}}{(s_1+ir_2)^{2t+1}}=(-)^t\Sigma_{-\infty}^{\infty}\Sigma_{-\infty}^{\infty}\frac{(-1)^{\frac{1}{2}(r_1-1)}i}{(s_2+ir_1)^{2t+1}}=\frac{\psi_{t-1}(1,2)\,\omega^{2t+1}}{(2t)!\,2^{2t+\frac{1}{2}}}$$

$$=\frac{2\pi^{2t+1}}{(2t)!}\,\Sigma_1^{\infty}E'_{2t}(n)\,e^{-n\pi}$$

$$=\frac{\pi^{2t+1}}{(2t)!\,2^{2t+1}}\{\psi_{t-1}(1,1)+(-)^t\,4\Sigma_1^{\infty}E_{2t}(n)\,e^{-n\pi}\},$$

$$\Sigma_{-\infty}^{\infty}\Sigma_{-\infty}^{\infty}\frac{(-1)^{\frac{1}{2}(r_2+s_1-1)}}{(s_1+ir_2)^{2t+1}}=(-)^t\Sigma_{-\infty}^{\infty}\Sigma_{-\infty}^{\infty}\frac{(-1)^{\frac{1}{2}(s_2+r_1-1)}i}{(s_2+ir_1)^{2t+1}}=\frac{\psi_{t-1}(2,1)\omega^{2t+1}}{(2t)!\,2^{2t}}$$

$$=\frac{\pi^{2t+1}}{(2t)!\,2^{2t-1}}\,\Sigma_1^{\infty}(-)^M E_{2t}(m)\,e^{-\frac{1}{2}m\pi}$$

$$=\frac{\pi^{2t+1}}{(2t)!\,2^{2t+1}}\{\psi_{t-1}(1,1)+(-)^{t-1}\,4\Sigma_1^{\infty}(-)^{n-1}E_{2t}(n)\,e^{-n\pi}\}.$$

For the sake of brevity in the above equalities only the expanded values of the q-series in powers of q (or the arithmetical* q-series) have been included. The equivalent unexpanded (or algebraical) forms may be obtained at once by means of the table in § 73, or from the more general formulæ in § 6 (pp. 4, 5): the identities arising from the equality of some of the different q-series in the above system are given in their algebraic form in § 94. The values of $F_t(1,1)$, $\chi_t(1,1)$, &c. in terms of Bernoullian or Eulerian numbers were given in § 74.

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* The expanded forms of the q-series in powers of q may be conveniently termed the arithmetical q-series, as their coefficients are arithmetical quantities dependent upon the divisors of the exponent. The arithmetical form is unique, but there are various algebraical forms (*i. e.* forms in which the terms are algebraical functions of q).

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§ 83. The values of the coefficients of the powers of ω as far as $t=5$ are readily derivable from the values of the functions given in § 30 (p. 24). They are as follows:

$$f_0(1,-1)=2, \qquad f_0(1,2)=2,$$
$$f_1(1,-1)=0, \qquad f_1(1,2)=2^3.3,$$
$$f_2(1,-1)=-2^4.9, \qquad f_2(1,2)=2^6.9,$$
$$f_3(1,-1)=0, \qquad f_3(1,2)=2^7.189,$$
$$f_4(1,-1)=2^9.189, \qquad f_4(1,2)=2^{13}.189,$$
$$f_5(1,-1)=0, \qquad f_5(1,2)=2^{11}.68607;$$

$$\phi_0(1,-1)=1, \qquad \phi_0(1,2)=1,$$
$$\phi_1(1,-1)=0, \qquad \phi_1(1,2)=3,$$
$$\phi_2(1,-1)=-12, \qquad \phi_2(1,2)=33,$$
$$\phi_3(1,-1)=0, \qquad \phi_3(1,2)=819,$$
$$\phi_4(1,-1)=3024, \qquad \phi_4(1,2)=34209,$$
$$\phi_5(1,-1)=0, \qquad \phi_5(1,2)=2189187;$$

$$\chi_1(1,-1)=0, \qquad \chi_1(1,2)=\tfrac{1}{2},$$
$$\chi_2(1,-1)=\tfrac{3}{5}, \qquad \chi_2(1,2)=-\tfrac{3}{20},$$
$$\chi_3(1,-1)=0, \qquad \chi_3(1,2)=\tfrac{3}{2},$$
$$\chi_4(1,-1)=\tfrac{42}{5}, \qquad \chi_4(1,2)=\tfrac{21}{40},$$
$$\chi_5(1,-1)=0, \qquad \chi_5(1,2)=\tfrac{189}{2};$$

$$\psi_0(1,-1)=1, \qquad \psi_0(1,2)=1,$$
$$\psi_1(1,-1)=-3, \qquad \psi_1(1,2)=9,$$
$$\psi_2(1,-1)=-27, \qquad \psi_2(1,2)=153,$$
$$\psi_3(1,-1)=441, \qquad \psi_3(1,2)=4977,$$
$$\psi_4(1,-1)=11529, \qquad \psi_4(1,2)=261009,$$
$$\psi_5(1,-1)=442827, \qquad \psi_5(1,2)=20039481;$$

$$\psi_0(2,1)=1,$$
$$\psi_1(2,1)=6,$$
$$F_2(1,-1)=\tfrac{2}{5}, \qquad \psi_2(2,1)=108,$$
$$F_3(1,-1)=0, \qquad \psi_3(2,1)=3528,$$
$$F_4(1,-1)=\tfrac{48}{5}, \qquad \psi_4(2,1)=184464,$$
$$F_5(1,-1)=0, \qquad \psi_5(2,1)=14170464.$$

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§ 84. All the equalities in § 82 hold good when $t=1$ with the exception of those in which the functions F_t and f_{t-2} occur. Corresponding to these excepted cases we find, from § 75, if the r-limits are infinitely larger than the s-limits,

$$-\Sigma_{-\infty}^{\infty}\Sigma_{-\infty}^{\infty}\frac{1}{(s_2+ir_2)^2}=-\Sigma_{-\infty}^{\infty}\Sigma_{-\infty}^{\infty}\frac{1}{(s_1+ir_1)^2}=\tfrac{1}{4}\pi$$

$$=2\pi^2\{\tfrac{1}{24}-\Sigma_1^{\infty}\sigma(n)\,e^{-2n\pi}\}=2\pi^2\Sigma_1^{\infty}(-)^{n-1}\Delta'(n)\,e^{-n\pi},$$

$$-\Sigma_{-\infty}^{\infty}\Sigma_{-\infty}^{\infty}\frac{1}{(s_2+ir_1)^2}=\tfrac{1}{4}(\omega^2+\pi)=2\pi^2\{\tfrac{1}{8}+\Sigma_1^{\infty}\zeta(n)\,e^{-2n\pi}\},$$

$$\Sigma_{-\infty}^{\infty}\Sigma_{-\infty}^{\infty}\frac{1}{(s_1+ir_2)^2}=\tfrac{1}{4}(\omega^2-\pi)=2\pi^2\Sigma_1^{\infty}\Delta'(n)\,e^{-n\pi}.$$

If the s-limits are infinitely larger than the r-limits the signs of the four double summations are to be changed, and in addition the values of the last two, which relate to the uneven numbers, are to be transposed.

§ 85. Corresponding to the case $t=0$ of the equalities in § 82, we have from § 66 or § 76, if the r-limits are infinitely larger than the s-limits,

$$\Sigma_{-\infty}^{\infty}\Sigma_{-\infty}^{\infty}\frac{(-1)^{\frac{1}{2}(r_1-1)}i}{s_2+ir_1}=\Sigma_{-\infty}^{\infty}\Sigma_{-\infty}^{\infty}\frac{(-1)^{\frac{1}{2}(s_1-1)}}{s_1+ir_2}+(-1)^{\frac{1}{2}(v_1+1)}\tfrac{1}{2}\pi$$

$$=\frac{\omega}{\sqrt{2}}=2\pi\{\tfrac{1}{4}+\Sigma_1^{\infty}E_0(n)\,e^{-n\pi}\},$$

$$\Sigma_{-\infty}^{\infty}\Sigma_{-\infty}^{\infty}\frac{(-1)^{\frac{1}{2}(s_2+r_1-1)}i}{s_2+ir_1}=\Sigma_{-\infty}^{\infty}\Sigma_{-\infty}^{\infty}\frac{(-1)^{\frac{1}{2}(s_1-1)}}{s_1+ir_1}+(-1)^{\frac{1}{2}(v_1+1)}\tfrac{1}{2}\pi$$

$$=\Sigma_{-\infty}^{\infty}\Sigma_{-\infty}^{\infty}\frac{(-1)^{\frac{1}{2}(r_2+s_1-1)}}{s_1+ir_2}=\Sigma_{-\infty}^{\infty}\Sigma_{-\infty}^{\infty}\frac{(-1)^{\frac{1}{2}(r_1-1)}i}{s_1+ir_1}$$

$$=\tfrac{1}{2}\omega=2\pi\{\tfrac{1}{4}+\Sigma_1^{\infty}(-)^n E_0(n)\,e^{-n\pi}\}=2\pi\Sigma_0^{\infty}E_0(4n+1)\,e^{-\frac{1}{2}(4n+1)\pi}.$$

§ 86. Since $F_t(1,-1)$, $f_t(1,-1)$, $\chi_t(1,-1)$, and $\phi_t(1,-1)$ vanish for uneven values of t, it follows from § 82, by replacing t by $2t+1$ or $2t$, that

$$\Sigma_{-\infty}^{\infty}\Sigma_{-\infty}^{\infty}\frac{1}{(s_2+ir_2)^{4t+2}}=0,\qquad \Sigma_{-\infty}^{\infty}\Sigma_{-\infty}^{\infty}\frac{1}{(s_1+ir_1)^{4t+2}}=0,$$

$$\Sigma_{-\infty}^{\infty}\Sigma_{-\infty}^{\infty}\frac{(-1)^{\frac{1}{2}(r_2+s_2)}}{(s_2+ir_2)^{4t+2}}=0,\qquad \Sigma_{-\infty}^{\infty}\Sigma_{-\infty}^{\infty}\frac{(-1)^{\frac{1}{2}(r_1+s_1)}}{(s_1+ir_1)^{4t}}=0.$$

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It is easy to see that these double summations must necessarily be zero if the r-limits and the s-limits are equal, *i.e.* if the area over which the summations extend is a square.

For, taking the first of the four expressions and denoting the limits of r_2 or s_2 by $\pm w_2$, then, whether w_2 be finite or infinite,

$$\Sigma_{-w_2}^{w_2}\Sigma_{-w_2}^{w_2}\frac{1}{(s_2+ir_2)^{4i+2}} = \Sigma_{-w_2}^{w_2}\Sigma_{-w_2}^{w_2}\frac{i^{4i+2}}{(r_2-is_2)^{4i+2}}$$

$$= -\Sigma_{-w_2}^{w_2}\Sigma_{-w_2}^{w_2}\frac{1}{(s_2+ir_2)^{4i+1}},$$

the first result being obtained by multiplying the numerator and denominator by i^{4i+2} and the second by changing the sign of i and interchanging r_2 and s_2. The value of the double summation, being unaffected by a change of sign, is therefore equal to zero.

If the exponent >2, it is immaterial whether the area of summation be a square or not, the values of the four double summations being independent of the ratio of the r-limits to the s-limits.

This is also true when the exponent is 2 in the case of the third summation, in which the terms are not all positive. But it is not so in the case of the first two summations, for it was shown in § 84 that

$$-\Sigma_{-v_2}^{v_2}\Sigma_{-u_2}^{u_2}\frac{1}{(s_2+ir_2)^2} = \Sigma_{-v_1}^{v_1}\Sigma_{-u_1}^{u_1}\frac{1}{(s_1+ir_1)^2} = \pm\tfrac{1}{4}\pi,$$

the upper or lower sign being taken according as the r-limits are infinitely larger than the s-limits or *vice versâ*.

Taking the first summation, and supposing u_2 infinitely larger than v_2, since

$$\Sigma_{-v_2}^{v_2}\Sigma_{-v_2}^{v_2}\frac{1}{(s_2+ir_2)^2} = 0,$$

it follows that

$$\Sigma\Sigma\frac{1}{(s_2+ir_2)^2} = \tfrac{1}{4}\pi,$$

if the limits of s_2 are $\pm v_2$ and the admissible values of r_2 extend from v_2 to u_2 and from $-v_2$ to $-u_2$, viz. if the area of summation consist of the two rectangles obtained by omitting a square of side $2v_2$ from the middle of an infinite rectangle of breadth $2v_2$ and length $2u_2$. This result may

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be expressed in the form

$$\Sigma_0^{v_2}\Sigma_{v_2}^{u_2}\frac{s_2^2-r_2^2}{(s_2^2+r_2^2)^2}=\tfrac{1}{16}\pi.$$

Similar remarks apply also to the second summation in which s_1+ir_1 occurs in place of s_2+ir_2.

Analogy with the formulæ for the sums of reciprocals of ordinary numbers, § 87.

§ 87. The results obtained in §§ 82, 84, 85 are analogous to those relating to the sums of powers of reciprocals of numbers in the ordinary theory. The method employed is also exactly similar to that by which the latter results are derived from the theory of circular functions.

For in the theory of circular functions we have (§§ 31 and 48)

$$\cot x=\Sigma_{-\infty}^{\infty}\frac{1}{x+\frac{1}{2}r_2\pi}=\frac{1}{x}+\Sigma_1^{\infty}(-)^t\chi_t(1,1)\frac{x^{2t-1}}{(2t-1)!},$$

$$\operatorname{cec}x=\Sigma_{-\infty}^{\infty}\frac{(-1)^{\frac{1}{2}r_2}}{x+\frac{1}{2}r_2\pi}=\frac{1}{x}+\Sigma_1^{\infty}\chi_t(1,0)\frac{x^{2t-1}}{(2t-1)!},$$

$$\tan x=-\Sigma_{-\infty}^{\infty}\frac{1}{x+\frac{1}{2}r_1\pi}=\Sigma_1^{\infty}\phi_{t-1}(1,1)\frac{x^{2t-1}}{(2t-1)!},$$

$$\sec x=\Sigma_{-\infty}^{\infty}\frac{(-1)^{\frac{1}{2}(r_1-1)}}{x+\frac{1}{2}r_1\pi}=1+\Sigma_1^{\infty}\psi_{t-1}(1,1)\frac{x^{2t}}{(2t)!};$$

whence, by equating the coefficients of x^{2t-1} and x^{2t},

$$2^{2t}\Sigma_{-\infty}^{\infty}\frac{1}{r_2^{2t}}=(-)^{t-1}\frac{\chi_t(1,1)}{(2t-1)!}\pi^{2t}=2^{2t}\frac{B_t}{(2t)!}\pi^{2t},$$

$$2^{2t}\Sigma_{-\infty}^{\infty}\frac{(-1)^{\frac{1}{2}r_2}}{r_2^{2t}}=-\frac{\chi_t(1,0)}{(2t-1)!}\pi^{2t}=-2(2^{2t-1}-1)\frac{B_t}{(2t)!}\pi^{2t},$$

$$2^{2t}\Sigma_{-\infty}^{\infty}\frac{1}{r_1^{2t}}=\frac{\phi_{t-1}(1,1)}{(2t-1)!}\pi^{2t}=2^{2t}(2^{2t}-1)\frac{B_t}{(2t)!}\pi^{2t},$$

$$2^{2t+1}\Sigma_{-\infty}^{\infty}\frac{(-1)^{\frac{1}{2}(r_1-1)}}{r_1^{2t+1}}=\frac{\psi_{t-1}(1,1)}{(2t)!}\pi^{2t+1}=\frac{E_t}{(2t)!}\pi^{2t+1};$$

the value $r_2=0$ being excluded from the first two series.

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We may write the above series in the more usual forms

$$1+\frac{1}{2^{2t}}+\frac{1}{3^{2t}}+\frac{1}{4^{2t}}+\&c.=2^{2t-1}\frac{B_t}{(2t)!}\pi^{2t},$$

$$1-\frac{1}{2^{2t}}+\frac{1}{3^{2t}}-\frac{1}{4^{2t}}+\&c.=(2^{2t-1}-1)\frac{B_t}{(2t)!}\pi^{2t},$$

$$1+\frac{1}{3^{2t}}+\frac{1}{5^{2t}}+\frac{1}{7^{2t}}+\&c.=\tfrac{1}{2}(2^{2t}-1)\frac{B_t}{(2t)!}\pi^{2t},$$

$$1-\frac{1}{3^{2t}}+\frac{1}{5^{2t}}-\frac{1}{7^{2t}}+\&c.=\tfrac{1}{2}\frac{E_t}{(2t)!}(\tfrac{1}{2}\pi)^{2t}.$$

Thus from the theory of the circular functions we obtain the sum of four series of reciprocals of numbers in three of which the exponents are even and in one uneven. The values of the three former depend upon the Bernoullian numbers and that of the latter upon Eulerian numbers.

In the case of the complex numbers there are ten series with even exponents whose values are expressed by means of the functions ϕ, f, F, χ (corresponding to the Bernoullian numbers) and six with even exponents whose values are expressed by means of the function ψ (corresponding to the Eulerian numbers).

The results in § 85 correspond to the formula

$$\Sigma_{-\infty}^{\infty}(-)^{\frac{1}{2}(r_1-1)}\frac{1}{r_1}=\tfrac{1}{2}\pi,$$

or, writing the series at length, to

$$1-\tfrac{1}{3}+\tfrac{1}{5}-\tfrac{1}{7}+\&c.=\tfrac{1}{4}\pi.$$

Formulæ for powers of ω as series proceeding by powers of $e^{-\pi}$, § 88.

§ 88. The equalities contained in § 82 afford expressions for the ratio of ω^t to π^t as series proceeding by powers of $e^{-\pi}$. If we denote the ratio $\frac{\omega}{\pi}$ by λ*, we thus obtain the following formulæ for λ^{2t} and λ^{2t+1}.

* Thus λ denotes the ratio of the total length of the lemniscate to that of the circle described upon the same diameter; see note, § 80 (p. 70).

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$$F_t(1,-1)\lambda^{2t} = F_t(1,1) + 2^{2t+1}\Sigma_1^\infty \sigma_{2t-1}(n)\, e^{-2n\pi};$$

$$f_{t-2}(1,-1)\lambda^{2t} = 2^{2t+1}\Sigma_1^\infty (-)^{n-1}\Delta'_{2t-1}(n)\, e^{-n\pi},$$

$$2f_{t-2}(1,2)\lambda^{2t} = f_{t-2}(1,1) + (-)^{t-1}2^{2t+1}\Sigma_1^\infty \zeta_{2t-1}(n)\, e^{-2n\pi}$$
$$= 2^{2t+1}\Sigma_1^\infty \Delta'_{2t-1}(n)\, e^{-n\pi};$$

$$\chi_t(1,-1)\lambda^{2t} = \chi_t(1,0) + (-)^t 4\Sigma_1^\infty (-)^{n-1}\Delta_{2t-1}(n)\, e^{-n\pi},$$

$$\chi_t(1,2)\lambda^{2t} = \chi_t(1,1) + 2^{2t+1}\Sigma_1^\infty \zeta'_{2t-1}(n)\, e^{-2n\pi}$$
$$= \chi_t(1,0) + (-)^{t-1}4\Sigma_1^\infty \Delta_{2t-1}(n)\, e^{-n\pi};$$

$$\phi_{t-1}(1,-1)\lambda^{2t} = 4\Sigma_1^\infty (-)^M \Delta_{2t-1}(m)\, e^{-\frac{1}{2}m\pi},$$

$$\sqrt{2}.\phi_{t-1}(1,2)\lambda^{2t} = \phi_{t-1}(1,1) + (-)^t 2^{2t+1}\Sigma_1^\infty \xi_{2t-1}(n)\, e^{-2n\pi}$$
$$= 4\Sigma_1^\infty \Delta_{2t-1}(m)\, e^{-\frac{1}{2}m\pi};$$

$$\psi_{t-1}(1,-1)\lambda^{2t+1} = 4\Sigma_1^\infty E_{2t}(m)\, e^{-\frac{1}{2}m\pi}$$
$$= (-)^{t-1}2^{2t+2}\Sigma_1^\infty (-)^{n-1}E'_{2t}(n)\, e^{-n\pi},$$

$$\sqrt{2}.\psi_{t-1}(1,2)\lambda^{2t+1} = \psi_{t-1}(1,1) + (-)^t 4\Sigma_1^\infty E_{2t}(n)\, e^{-n\pi}$$
$$= 2^{2t+2}\Sigma_1 E'_{2t}(n)\, e^{-n\pi},$$

$$2\psi_{t-1}(2,1)\lambda^{2t+1} = \psi_{t-1}(1,1) + (-)^{t-1}4\Sigma_1^\infty (-)^{n-1}E_{2t}(n)\, e^{-n\pi}$$
$$= 4\Sigma_1^\infty (-)^M E_{2t}(m)\, e^{-\frac{1}{2}m\pi}.$$

Identical equations derived from the preceding formulæ, §§ 89, 90.

§ 89. In six of the preceding ten formulæ there occur two different series. The equality of these series gives rise to a set of six identities which may be written as follows:

$$4\Sigma_1^\infty \Delta_{2t-1}(2n)\, e^{-n\pi} + (-)^t 2^{2t+1}\Sigma_1^\infty \zeta'_{2t-1}(n)\, e^{-2n\pi}$$
$$= (-)^{t-1}\{\chi_t(1,1) - \chi_t(1,0)\},$$

$$4\Sigma_1^\infty \Delta'_{2t-1}(2n)\, e^{-n\pi} + (-)^t 2^{2t+1}\Sigma_1^\infty \zeta_{2t-1}(n)\, e^{-2n\pi} = f_{t-2}(1,1);$$

$$4\Sigma_1^\infty \Delta_{2t-1}(m)\, e^{-\frac{1}{2}m\pi} - (-)^t 2^{2t+1}\Sigma_1^\infty \xi_{2t-1}(n)\, e^{-2n\pi} = \phi_{t-1}(1,1);$$

$$4\Sigma_1^\infty E'_{2t}(2n)\, e^{-n\pi} - (-)^t 4\Sigma_1^\infty E_{2t}(2n)\, e^{-n\pi} = \psi_{t-1}(1,1),$$

$$4\Sigma_1^\infty (-)^M E_{2t}(m)\, e^{-\frac{1}{2}m\pi} + (-)^t 4\Sigma_1^\infty (-)^{n-1}E_{2t}(2n)\, e^{-n\pi} = \psi_{t-1}(1,1),$$

$$4\Sigma_1^\infty E_{2t}(m)\, e^{-\frac{1}{2}m\pi} + (-)^t 4\Sigma_1^\infty (-)^{n-1}E'_{2t}(2n)\, e^{-n\pi} = 0.$$

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The values of the right-hand members of the first five equations in terms of Bernoullian and Eulerian numbers are as follows (§ 31):

$$(-)^{t-1}\{\chi_t(1,1)-\chi_t(1,0)\}=2^{2t-1}\frac{B_t}{t}+(-)^t(2^{2t-1}-1)\frac{B_t}{t},$$

$$f_{t-2}(1,1)=\phi_{t-1}(1,1)=2^{2t-1}(2^{2t}-1)\frac{B_t}{t},$$

$$\psi_{t-1}(1,1)=E_t.$$

§ 90. Also, since $F_t(1,-1)$, $\chi_t(1,-1)$, and $f_{t-2}(1,-1)$ vanish when t is uneven, and $\phi_{t-1}(1,-1)$ vanishes when t is even, we find, from § 88 that, if t be uneven,

$$2^{2t+1}\Sigma_1^\infty \sigma_{2t-1}(n)\,e^{-2n\pi} = -F_t(1,1)=2^{2t-1}\frac{B_t}{t},$$

$$4\Sigma_1^\infty(-)^{n-1}\Delta_{2t-1}(n)\,e^{-n\pi} = \chi_t(1,0)=(2^{2t-1}-1)\frac{B_t}{t},$$

$$\Sigma_1^\infty(-)^{n-1}\Delta'_{2t-1}(n)\,e^{-n\pi}=0;$$

and, if t be even,

$$\Sigma_1^\infty(-)^M\Delta_{2t-1}(m)\,e^{-\frac{1}{2}m\pi}=0.$$

Series for Bernoullian and Eulerian Numbers, §§ 91–94.

§ 91. By considering separately the cases of t uneven and t even in the first two formulæ of § 89, we may derive the formulæ*

$$\frac{B_t}{t}=4\Sigma_1^\infty(-)^{n-1}\zeta'_{2t-1}(n)\,e^{-n\pi} \quad (t \text{ uneven}),$$

$$(2^{2t}-1)\frac{B_t}{t}=4\Sigma_1^\infty(-)^{n-1}\zeta_{2t-1}(n)\,e^{-n\pi} \quad (t \text{ uneven}),$$

$$(2^{2t}-1)\frac{B_t}{t}=4\Sigma_1^\infty(-)^{n-1}\xi_{2t-1}(n)\,e^{-n\pi} \quad (t \text{ even}).$$

* In obtaining these formulæ the following relations which may be derived from § 54 are required:

$$\zeta'_r(2n)+\Delta_r(2n)=2^r\zeta'_r(n), \quad \zeta_r(2n)+\Delta'_t(2n)=\zeta_t(n).$$

The first formula (*i.e.* that involving ζ) may also be deduced from § 90.

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We also find, from § 90,

$$\frac{B_t}{t} = 4\Sigma_1^\infty \sigma_{2t-1}(n)\, e^{-2n\pi} \qquad (t \text{ uneven}),$$

$$\frac{B_t}{t} = 4\Sigma_1^\infty (-)^n \sigma_{2t-1}(n)\, e^{-n\pi} \qquad (t \text{ uneven}),$$

$$(2^{2t-1}-1)\frac{B_t}{t} = 4\Sigma_1^\infty (-)^{n-1} \Delta_{2t-1}(n)\, e^{-n\pi} \qquad (t \text{ uneven}).$$

§ 92. The third formula of § 89 may be written

$$(2^{2t}-1)\frac{B_t}{t} = 2^{-2t+3}\Sigma_1^\infty \Delta_{2t-1}(m)\, e^{-\frac{1}{2}m\pi} + (-)^{t-1} 4\Sigma_1^\infty \xi_{2t-1}(n)\, e^{-2n\pi}.$$

It may be noticed also that, from § 90, we may derive the formula

$$\frac{B_t}{t} = 2^{-2t+3}\Sigma_1^\infty \Delta_{2t-1}(m)\, e^{-m\pi} + 4\Sigma_1^\infty \sigma_{2t-1}(n)\, e^{-4n\pi}.$$

§ 93. From the last three formulæ in § 89 we deduce the following series for the Eulerian numbers:

$$E_t = 4\Sigma_1^\infty \{E'_{2t}(2n) + (-)^{t-1} E_{2t}(2n)\}\, e^{-n\pi},$$

$$E_t = 4\Sigma_1^\infty \pm E_{2t}(n)\, e^{-\frac{1}{2}n\pi}.$$

We also find

$$0 = \Sigma_1^\infty \pm E'_{2t}(n)\, e^{-\frac{1}{2}n\pi}.$$

In the last two series the law of the signs is $+--+$ or $++--$, according as t is uneven or even.

§ 94. The formulæ in § 91 are so curious that it seems interesting to write them also in their algebraical form, viz., we have, if t be uneven,

$$\frac{B_t}{t} = 4\Sigma_1^\infty \frac{n^{2t-2}}{e^{2n\pi}-1}$$

$$= 4\Sigma_1^\infty (-)^{n-1} \frac{n^{2t-2}}{e^{n\pi}+(-1)^n} = 4\Sigma_1^\infty (-)^n \frac{n^{2t-1}}{e^{n\pi}-(-1)^n},$$

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$$(2^{2t-1}-1)\frac{B_t}{t}=4\Sigma_1^\infty\frac{m^{2t-1}}{e^{m\pi}+1},$$

$$(2^{2t}-1)\frac{B_t}{t}=4\Sigma_1^\infty\frac{n^{2t-1}}{e^{n\pi}-(-1)^n};$$

and, if t be even,

$$(2^{2t}-1)\frac{B_t}{t}=4\Sigma_1^\infty\frac{n^{2t-1}}{e^{n\pi}+(-1)^n}.$$

Thus, for both positive and negative values of t,

$$(2^{2t}-1)\frac{B_t}{t}=4\Sigma_1^\infty\frac{n^{2t-1}}{e^{n\pi}+(-1)^{n+t}}.$$

It may be noticed also that, from § 90, we have the identities

$$\Sigma_1^\infty(-)^{n-1}\frac{n^{2t-1}}{e^{n\pi}-e^{-n\pi}}=0 \qquad (t \text{ uneven}),$$

$$\Sigma_1^\infty(-)^{\frac{1}{2}(m-1)}\frac{m^{2t-1}}{e^{\frac{1}{2}m\pi}+e^{-\frac{1}{2}m\pi}}=0 \quad (t \text{ even}).$$

Conclusion, § 95.

§ 95. It had been my intention to have added some results relating to the lemniscate functions, and to have concluded the paper by certain applications* of the formulæ in §§ 70 and 71. I omit these investigations, however, as the great length to which the paper has already extended renders it undesirable that I should occupy more space in the *Messenger* at present.

---

* The principal applications referred to relate to the representations of a number as a sum of 2, 4, 6, 8, or 12 squares, and to the proofs of the results contained in my papers "On the square of the series in which the coefficients are the sums of the divisors of the exponents," and "Expressions for the first five powers of the series in which the coefficients are the sums of the divisors of the exponents" (*Messenger*, vol. XIV., pp. 156–163, vol. XV., pp. 33–36, 1885).

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ERRATA.

p. 4, lines 8 and 9. The functional sign ϕ is omitted in three places, and x is printed for n in one place. The equations should be

$$\xi\phi(n) = \zeta\phi(n) + 2(-)^n \Delta'\phi(n)$$
$$= \zeta'\phi(n) + 2(-)^n \Delta\phi(n).$$

p. 19, line 9, *for* "$e^{(2n+1)}$" *read* "$e^{(2n+1)x}$."

p. 21, bottom line, *for* "F_{n-1}" *read* "F_{n+1}."

p. 22, line 7, *for* "k^2" *read* "h."

„ line 8, *for* "$f_{n-1}(h, h')$" *read* "$f_{n-1}(h, -h')$."

„ line 11, *for* "$(-)^{n-1}\psi_{n-1}(h, 1)$" *read* "$(-)^{n-1} h\psi_{n-1}(h, 1)$."

p. 25, line 7 from bottom, *for* "$(2^{n-1} - 1)$" *read* "$(2^{2n-1} - 1)$."

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p. 36. In the last eight formulæ the argument is printed x instead of ρx. In five of the last six formulæ the sign prefixed to the double summation is erroneous, viz. the signs should be

$$-, \quad +, \quad +, \quad -, \quad -, \quad +$$

(so that the sign is to be changed in the expressions for $k'\rho \operatorname{nd}\rho x$, $\rho \operatorname{sd}\rho x$, $k\rho \operatorname{sn}\rho x$, $k\rho \operatorname{cn}\rho x$, and $\rho \operatorname{zn}\rho x$). The signs are correct as printed in § 43.

The double summations in § 42 (pp. 35, 36) may be most readily deduced from § 20 (pp. 16, 17) by substituting for the hyperbolic functions their values as given by the formulæ

$$\coth x = i \sum_{-\infty}^{\infty} \frac{1}{\frac{1}{2}r_2\pi + ix}, \qquad \tanh x = i \sum_{-\infty}^{\infty} \frac{1}{\frac{1}{2}r_1\pi + ix},$$

$$\operatorname{cech} x = i \sum_{-\infty}^{\infty} (-)^r \frac{1}{\frac{1}{2}r_2\pi + ix}, \qquad \operatorname{sech} x = -\sum_{-\infty}^{\infty} (-)^r \frac{1}{\frac{1}{2}r_1\pi + ix},$$

and replacing m and $2n$ by s_1 and s_2 respectively.

The values of the hyperbolic functions given in § 42 are quite correct, but the above values lead more directly to the double summations in the form in which they are presented in § 42.

p. 47, between lines 7 and 8, *for* "$\zeta_s(n)$" *read* "$\xi_s(n)$."

p. 60, line 11 from bottom, *for* "ϕ_{t-1}" *read* "ψ_{t-1}."

p. 76, line 8, *for* "$)^{2t+1}$" *read* "$)^{2t+2}$."

The functional sign cosec has been throughout abbreviated into cec, in order that all the six circular functions might be represented by three-letter functional signs. This produces a greater regularity in the appearance of the systems of formulæ. For a similar reason the hyperbolic cosecant has been abbreviated from cosech to cech.

NOTE ON THE POTENTIAL OF AN ELLIPTIC CYLINDER.

By *W. Burnside.*

THE potential of an elliptic cylinder is usually obtained in the text-books by making c infinite in the expression for the potential of an ellipsoid. It is the object of this note to obtain it by direct and comparatively simple integration.

The variable part of the potential due to matter distributed with uniform line-density μ along an infinite straight line is $-2\mu \log r$ at distance r from the line, and the variable part of the potential of a cylinder of uniform volume density ρ at x, y, z is therefore

$$-\rho \iint \log\{(x'-x)^2 + (y'-y)^2\}\, dx'dy',$$

where x', y' are the coordinates of any point in the cross-sections of the cylinder through x, y, z and the integral is extended over the area of this cross-section. The axis of z is taken parallel to the generators.

Consider first a shell bounded by similar, similarly-situated, and coaxial elliptic cylinders, the axes of whose cross-sections are a', b' and $a'(1+\tau)$, $b'(1+\tau)$; τ being infinitesimal,

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The above integral may then be written

$$-\rho\tau \int \log\{(x'-x)^2+(y'-y)^2\}\, p'ds' \ldots\ldots\ldots (A),$$

where ds' is an element of arc of the elliptic boundary, and p' the perpendicular on it from the centre.

Putting $x' = a'\cos\phi', \quad y' = b'\sin\phi',$

$$x = a\cos\phi, \quad y = b\sin\phi,$$

then $$p'ds' = a'b'd\phi',$$

and $$(x'-x)^2+(y'-y)^2 = \tfrac{1}{2}(a^2+b^2+a'^2+b'^2) - 2aa'\cos\phi\cos\phi' - 2bb'\sin\phi\sin\phi' + \tfrac{1}{2}(a^2-b^2)\cos 2\phi + \tfrac{1}{2}(a'^2-b'^2)\cos 2\phi'.$$

Between a, b, ϕ one relation may be taken.

Put $a^2-b^2 = a'^2-b'^2$, then

$$(x'-x)^2+(y'-y)^2 = \tfrac{1}{2}(a^2+b^2+a'^2+b'^2) - (aa'+bb')\cos(\phi-\phi') - (aa'-bb')\cos(\phi+\phi') + (a^2-b^2)\cos(\phi-\phi')\cos(\phi+\phi'),$$

and it may be verified at once that this

$$= \tfrac{1}{2}(a^2+b^2+a'^2+b'^2)\{1-m\cos(\phi-\phi')\}\{1-n\cos(\phi+\phi')\},$$

where $$m(a^2+b^2+a'^2+b'^2) = 2(aa'+bb'),$$

$$n(a^2+b^2+a'^2+b'^2) = 2(aa'-bb').$$

The quantities m, n are both less than unity, so that the only integral now involved is of the form

$$\int_0^{2\pi} \log(1-2x\cos\phi+x^2)\, d\phi.$$

Moreover, by comparing this with the form (A) it is seen that if x be less than unity it is proportional to the variable part of the potential inside a thin hollow circular cylinder, and therefore by a known elementary theorem is independent of x; whence, putting zero for x, it vanishes.

The necessary algebra is made most simple by introducing the other elliptic co-ordinate; *i.e.* by putting

$$a' = c\cosh\alpha', \quad b' = c\sinh\alpha',$$

$$a = c\cosh\alpha, \quad b = c\sinh\alpha,$$

where $$c^2 = a^2-b^2,$$

whence $$m = \operatorname{sech}(\alpha-\alpha'), \quad n = \operatorname{sech}(\alpha+\alpha'),$$

$$\tfrac{1}{2}(a^2+b^2+a'^2+b'^2) = c^2\cosh(\alpha+\alpha')\cosh(\alpha-\alpha'),$$

and

$$(x'-x)^2+(y'-y)^2 = c^2\{\cosh(\alpha-\alpha')-\cos(\phi-\phi')\}\{\cosh(\alpha+\alpha')-\cos(\phi+\phi')\}.$$

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Now

$$\int_0^{2\pi} \log\{\cosh(\alpha+\alpha') - \cos(\phi+\phi')\}\, d\phi'$$

$$= \int_0^{2\pi} [\log\{1 - 2e^{-(\alpha+\alpha')}\cos(\phi+\phi') + e^{-2(\alpha+\alpha')}\} - \log 2e^{-(\alpha+\alpha)}]\, d\phi'$$

$$= 2\pi(\alpha+\alpha') - 2\pi\log 2 \ldots\ldots\ldots\

$$\text{Now } \int_0^{2\pi} \cos 2\phi' \log\{\cosh(\alpha+\alpha') - \cos(\phi+\phi')\}\, d\phi'$$

$$= \int_0^{2\pi} \cos 2\phi' \log\{1 - 2e^{-(\alpha+\alpha')}\cos(\phi+\phi') + e^{-2(\alpha+\alpha')}\}\, d\phi'$$

$$= -2\int_0^{2\pi} \cos 2\phi' . \sum_1^\infty \frac{1}{n} e^{-n(\alpha+\alpha')} \cos n(\phi+\phi')\, d\phi'$$

$$= -\tfrac{1}{2} e^{-2(\alpha+\alpha')} \int_0^{2\pi} \cos 2\phi\, d\phi'$$

$$= -\pi e^{-2(\alpha+\alpha')} \cos 2\phi.$$

Similarly, if $\alpha > \alpha'$,

$$\int_0^{2\pi} \cos 2\phi' \log\{\cosh(\alpha-\alpha') - \cos(\phi-\phi')\}\, d\phi' = -\pi e^{-2(\alpha-\alpha')} \cos 2\phi,$$

and hence the variable part of V

$$= -\tfrac{1}{2} c^2 \rho \int_0^{\alpha'} 2\pi \left[2\alpha \cosh 2\alpha' + e^{-2\alpha} \cos 2\phi \cosh 2\alpha'\right] d\alpha'$$

$$= -M\left[2\alpha + e^{-2\alpha} \cos 2\phi\right],$$

where M is the mass per unit length of the solid cylinder.

The transformation of the usual form for the potential of an elliptic cylinder to the above is made by Prof. Lamb at p. 33 of vol. VII of the *Messenger*.

The expressions for V obtained after the integration with respect to ϕ', shews at once that the external potential of a solid elliptic cylinder is the same as that of a hollow cylinder of equal mass, with the same external boundary and whose inner boundary is any cylinder confocal with the outer one.

For a thin shell bounded by similar hyperbolic cylinders, the integration is itself interesting. With a notation similar to that already used,

$$p'ds' = a'b'd\alpha',$$

and the required potential is

$$-\rho\tau a'b' \int \log\left[c^2\{\cosh(\alpha-\alpha') - \cos(\phi-\phi')\}\right.$$
$$\left. \times \{\cosh(\alpha+\alpha') - \cos(\phi+\phi')\}\right] d\alpha'.$$

The limits of the integral for the x-positive part of the hyperbola are $-\infty$ to $+\infty$. For the other half $\alpha' - \pi i$ must be written for α' and limits $+\infty$ to $-\infty$ be taken. The potential then depends on an integral of the form

$$\int_\infty^\infty \log\left[(\cosh\alpha' - \cos\theta)/(\cosh\alpha' + \cos\theta)\right] d\alpha'.$$

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By expanding the logarithm in powers of $e^{-\alpha'}$ it may be easily verified that the value of this integral is $2\pi(\theta - \pi/2)$, where θ is the least positive angle corresponding to the given cosine.

Hence, again omitting an infinite constant, the variable part of the potential for the hyperbolic cylindrical shell is, when $\phi > \phi'$.

$$-2\pi\rho\tau a'b'[\phi - \phi' - \pi/2 + \phi + \phi' - \pi/2],$$

or
$$-2\pi\rho\tau a'b'[2\phi - \pi].$$

If $\phi > \phi'$, that is for points separated by one part of the shell from the other the factor in the bracket is $2\phi' - \pi$, and the potential again is constant, as it should be.

In these latter expressions, ϕ is to be regarded as the same same for both branches of an hyperbola.

It may be remarked that for densties varying according to any rational integral function of the co-ordinates the integrations to be performed are hardly more laborious than those given.

THE LINES OF CURVATURE ON A PARALLEL SURFACE TO A QUADRIC.

By *J. P. Johnston.*

LET $\frac{x^2}{a} + \frac{y^2}{b} + \frac{z^2}{c} = 1$ be the equation of the quadric, and k the length of the common intercept on the normals of the quadric by the parallel surface.

It is well known that the normals along any line of curvature on the quadric intersect the parallel surface along a line of curvature.

Let x', y', z', be the coordinates of a point on the line of curvature traced out by the intersection of the quadric whose equation is $\frac{x^2}{a+\lambda} + \frac{y^2}{b+\lambda} + \frac{z^2}{c+\lambda} = 1$ with the given quadric. Then since the direction cosines of the normal at $x'y'z'$ are $\frac{px'}{a}, \frac{py'}{b}, \frac{pz'}{c}$, where p is the length of the perpendicular from the centre of the given quadric on the tangent plane to it at $x'y'z'$, the corresponding point (x, y, z) on the parallel surface is given by the equations

$$x = x'\left(1 + \frac{pk}{a}\right), \quad y = y'\left(1 + \frac{pk}{b}\right), \quad z = z'\left(1 + \frac{pk}{c}\right),$$

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x', y', z', and p being connected by the relations

$$\frac{x'^2}{a}+\frac{y'^2}{b}+\frac{z'^2}{c}=1,$$

$$\frac{x'^2}{a+\lambda}+\frac{y'^2}{b+\lambda}+\frac{z'^2}{c+\lambda}=1,$$

$$\frac{x'^2}{a^2}+\frac{y'^2}{b^2}+\frac{z'^2}{c^2}=\frac{1}{p^2}.$$

Solving these equations for x'^2, y'^2, z'^2, we get

$$x'^2=-\frac{a^2(a+\lambda)}{(c-a)(a-b)}\left(1+\frac{bc}{\lambda p^2}\right),$$

$$y'^2=-\frac{b^2(b+\lambda)}{(a-b)(b-c)}\left(1+\frac{ca}{\lambda p^2}\right),$$

$$z'^2=-\frac{c^2(c+\lambda)}{(b-c)(c-a)}\left(1+\frac{ab}{\lambda p^2}\right).$$

Therefore

$$x^2=-\frac{a^2(a+\lambda)}{(c-a)(a-b)}\left(1+\frac{bc}{\lambda p^2}\right)\left(1+\frac{pk}{a}\right)^2,$$

with similar values for y^2 and z^2.

Therefore

$$\frac{x^2}{a+\lambda}=\frac{-1}{(c-a)(a-b)}\left\{\frac{a^2bc}{\lambda p^2}+\frac{2abck}{\lambda p}+a^2+\frac{bck^2}{\lambda}+2apk+p^2k^2\right\},$$

with similar values for $\frac{y^2}{b+\lambda}$ and $\frac{z^2}{c+\lambda}$.

But $$\Sigma\frac{1}{(c-a)(a-b)}=0, \quad \Sigma\frac{a}{(c-a)(a-b)}=0,$$

$$\Sigma\frac{bc}{(c-a)(a-b)}=-1, \quad \Sigma\frac{a^2}{(c-a)(a-b)}=-1.$$

Therefore $$\frac{x^2}{a+\lambda}+\frac{y^2}{b+\lambda}+\frac{z^2}{c+\lambda}=1+\frac{k^2}{\lambda}\ldots\ldots\ldots\ldots(\alpha).$$

Consequently the lines of curvature on a surface parallel to a quadric lie on a series of quadrics coaxial with the given quadric.

Since the square is the only power of k which enters the equation of the quadric (α), this quadric cuts both sheets of the parallel surface in lines of curvature.

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NOTE ON THE DIFFERENTIAL EQUATION $\frac{dx}{\sqrt{(1-x^4)}} + \frac{dy}{\sqrt{(1-y^4)}} = 0.$

By Prof. *Cayley*.

We have

$$\frac{\sin u - \sin v}{\cos u - \cos v} = -\cot \tfrac{1}{2}(u+v), = -\sqrt{\left\{\frac{1+\cos(u+v)}{1-\cos(u+v)}\right\}},$$

and thence, writing $\cos u = x$, $\sin u = \sqrt{(1-x^2)} = \sqrt{(X)}$, and similarly $\cos v = y$, $\sin v = \sqrt{(1-y^2)} = \sqrt{(Y)}$, we have

$$\frac{\sqrt{(X)} - \sqrt{(Y)}}{x-y} = -\sqrt{\left\{\frac{1+xy-\sqrt{(XY)}}{1-xy+\sqrt{(XY)}}\right\}},$$

an identical equation which, in the form

$$\frac{2-x^2-y^2-2\sqrt{(XY)}}{(x-y)^2} = \frac{1+xy-\sqrt{(XY)}}{1-xy+\sqrt{(XY)}},$$

may be verified directly without any difficulty. The integral of the proposed differential equation can of course be taken to be $c = xy - \sqrt{(XY)}$; and we have thus another form of integral

$$\frac{\sqrt{(X)} - \sqrt{(Y)}}{x-y} = -\sqrt{\left(\frac{1+c}{1-c}\right)}, \text{ say } = \sqrt{(C)},$$

viz. we have the integral

$$\left\{\frac{\sqrt{(X)} - \sqrt{(Y)}}{x-y}\right\}^2 = C,$$

which is what Lagrange's integral of the differential equation

$$\frac{dx}{\sqrt{(X)}} + \frac{dy}{\sqrt{(Y)}} = 0$$

becomes when the quartic functions X, Y reduce themselves to the quadric functions $1-x^2$ and $1-y^2$ respectively.

NOTES ON INEQUALITIES.

By *F. C. Wace, M.A.*

The following proofs of certain well-known inequalities are much simpler than those found in the ordinary Text Books. I have given them in lectures for several years, but, so far as I know, they have never been published.

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1. $a_1^{a_1}.a_2^{a_2}\ldots a_n^{a_n} > \left(\frac{a_1+a_2\ldots+a_n}{n}\right)^{a_1+a_2\ldots+a_n}$ .....(*A*).

Let there be a_1 things each equal to $\frac{1}{a_1}$,

a_2 ..... $\frac{1}{a_2}$,

.....,

a_n ..... $\frac{1}{a_n}$.

The arithmetical mean of all these is

$$\frac{a_1\frac{1}{a_1}+a_2\frac{1}{a_2}\ldots+a_n\frac{1}{a_n}}{a_1+a_2\ldots+a_n} \text{ or } \frac{n}{a_1+a_2\ldots+a_n}.$$

Their geometrical mean is

$$\left\{\left(\frac{1}{a_1}\right)^{a_1}\left(\frac{1}{a_2}\right)^{a_2}\ldots\left(\frac{1}{a_n}\right)^{a_n}\right\}^{\frac{1}{a_1+a_2\ldots+a_n}}.$$

The former of these is the greater, therefore

$$\left(\frac{n}{a_1+a_2\ldots+a_n}\right)^{a_1+a_2\ldots+a_n} > \frac{1}{a_1^{a_1}}\frac{1}{a_2^{a_2}}\ldots\frac{1}{a_n^{a_n}},$$

or $$a_1^{a_1}.a_2^{a_2}\ldots a_n^{a_n} > \left\{\frac{a_1+a_2\ldots+a_n}{n}\right\}^{a_1+a_2\ldots+a_n}.$$

2. The proposition on which the above proof depends, viz. that the arithmetical mean of n quantities is never less than their geometrical mean, seems to imply that n is an integer, and hence to require that $a_1, a_2, \ldots, a_n$ must be integers. This restriction may be removed as follows. If $a_1, a_2, \ldots, a_n$ be not all integers, let m be such a quantity that $ma_1, ma_2, \ldots, ma_n$ are integral, then the inequality (*A*) is true for $ma_1, ma_2, \ldots, ma_n$, *i.e.*

$$(ma_1)^{ma_1}.(ma_2)^{ma_2}\ldots(ma_n)^{ma_n} > \left\{\frac{ma_1+ma_2\ldots+ma_n}{n}\right\}^{ma_1+\ldots+ma_n},$$

or, extracting m^{th} root,

$$m^{a_1+a_2\ldots+a_n}.a_1^{a_1}.a_2^{a_2}\ldots a_n^{a_n} > \left\{\frac{m(a_1+a_2\ldots+a_n)}{n}\right\}^{a_1+a_2\ldots+a_n};$$

therefore

$$a_1^{a_1}.a_2^{a_2}\ldots a_n^{a_n} > \left\{\frac{a_1+a_2\ldots+a_n}{n}\right\}^{a_1+a_2\ldots+a_n}.$$

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The general proposition, or any application of it, may be easily extended to the case of non-integral quantities in a similar manner.

3. $n^p\{a_1^{p+1}+a_2^{p+1}\ldots+a_n^{p+1}\}>\{a_1+a_2\ldots+a_n\}^{p+1}\ldots\ldots(B)$,

if p be positive, or negative and greater than 1.

Let there be a_1 quantities each equal to a_1^p,

a_2 ..... a_2^p,

.....,

a_n ..... a_n^p.

Their arithmetical mean is

$$\frac{a_1^{p+1}+a_2^{p+1}\ldots+a_n^{p+1}}{a_1+a_2\ldots+a_n}.$$

Their geometrical mean is

$$\{a_1^{pa_1}.a_2^{pa_2}\ldots a_n^{pa_n}\}^{\frac{1}{a_1+a_2\ldots+a_n}},$$

or

$$\{a_1^{a_1}.a_2^{a_2}\ldots a_n^{a_n}\}^{\frac{p}{a_1+a_2\ldots+a_n}};$$

therefore

$$\frac{a_1^{p+1}+a_2^{p+1}+\ldots a_n^{p+1}}{a_1+a_2\ldots+a_n}>\{a_1^{a_1}.a_2^{a_2}\ldots a_n^{a_n}\}^{\frac{p}{a_1+a_2\ldots+a_n}}$$

$$>\left\{\frac{a_1+a_2\ldots+a_n}{n}\right\}^p,$$

by (A), if p be positive; therefore

$$n^p\{a_1^{p+1}+a_2^{p+1}\ldots+a_n^{p+1}\}>\{a_1+a_2\ldots+a_n\}^{p+1}.$$

4. If p be negative and less than 1, $p+1$ is positive and <1.

Let $p+1=\frac{1}{q+1}$; therefore $q+1$ is positive and >1; therefore B is true for q.

Hence

$$n^q\{b_1^{q+1}+b_2^{q+1}\ldots+b_n^{q+1}\}>\{b_1+b_2\ldots+b_n\}^{q+1}.$$

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Let $b_1^{q+1} = a_1, \quad b_2^{q+1} = a_2, \quad \ldots, \quad b_n^{q+1} = a_n;$

therefore $b_1 = a_1^{p+1}, \quad b_2 = a_2^{p+1}, \quad \ldots, \quad b_n = a_n^{p+1};$

therefore

$$n^q \{a_1 + a_2 \ldots + a_n\} > \{a_1^{p+1} + a_2^{p+1} \ldots + a_n^{p+1}\}^{q+1},$$

or $$a_1^{p+1} + a_2^{p+1} + a_n^{p+1} < n^{\frac{q}{q+1}} \{a_1 + a_2 \ldots + a_n\}^{\frac{1}{q+1}},$$

and $$\frac{q}{q+1} = 1 - \frac{1}{q+1} = 1 - (p+1) = -p;$$

therefore

$$n^p \{a_1^{p+1} + a_2^{p+1} \ldots + a_n^{p+1}\} < \{a_1 + a_2 \ldots + a_n\}^{p+1}.$$

5. If p be negative and greater than 1, let $p = -q$; therefore $q - 1$ is positive; then

$$n^p \{a_1^{p+1} + \ldots + a_n^{p+1}\}$$

$$= \frac{1}{n^q} \left\{\frac{1}{a_1^{q-1}} + \frac{1}{a_2^{q-1}} + \ldots + \frac{1}{a_n^{q-1}}\right\}$$

$$= \frac{1}{n^{2q-2}} n^{q-2} \left\{\left(\frac{1}{a_1}\right)^{q-1} + \left(\frac{1}{a_2}\right)^{q-1} + \ldots + \left(\frac{1}{a_n}\right)^{q-1}\right\}$$

$$> \frac{1}{n^{2q-2}} \left\{\frac{1}{a_1} + \frac{1}{a_2} \ldots + \frac{1}{a_n}\right\}^{q-1}, \text{ if } q-2 \text{ be positive.}$$

Now $$\{a_1 + a_2 \ldots + a_n\} \left\{\frac{1}{a_1} + \frac{1}{a_2} \ldots + \frac{1}{a_n}\right\} > n^2;$$

[For, on multiplying out, there are n terms, each $= 1$ and $\frac{1}{2}n(n-1)$ pairs of the form $\frac{a_r}{a_s} + \frac{a_s}{a_r}$, each of which is greater than 2] therefore

$$n^p \{a_1^{p+1} + \ldots + a_n^{p+1}\} > \frac{1}{n^{2q-2}} \left\{\frac{n^2}{a_1 + a_2 \ldots + a_n}\right\}^{q-1}$$

$$> \left\{\frac{1}{a_1 + a_2 \ldots + a_n}\right\}^{q-1}$$

$$> (a_1 + a_2 \ldots + a_n)^{p+1};$$

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but, if $q-2$ be negative,

$$n^p\{a_1^{p+1}\ldots+a_n^{p+1}\}=\frac{1}{n^q}\left\{\frac{1}{a_1^{q-1}}+\ldots+\frac{1}{a_n^{q-1}}\right\}$$

$$>\frac{1}{n^q}\frac{n^2}{a_1^{q-1}+\ldots+a_n^{q-1}},\text{ as above,}$$

$$>\frac{1}{n^{q-2}\{a_1^{q-1}+\ldots+a_n^{q-1}\}}$$

$$>\frac{1}{\{a_1+a_2\ldots+a_n\}^{q-1}},\text{ by (4),}$$

$$>\{a_1+a_2\ldots+a_n\}^{p+1}.$$

6. $$\frac{\{a_1^p+a_2^p+\ldots+a_n^p\}^q}{\{a_1^q+a_2^q+\ldots+a_n^q\}^p}>\frac{1}{n^{p-q}},$$

if p be greater than q.

Let there be a_1^q quantities each equal to a_1^{p-q},

a_2^q ..... a_2^{p-q},

.....,

a_n^q ..... a_n^{p-q}.

Then the arithmetical mean

$$=\frac{a_1^p+a_2^p\ldots+a_n^p}{a_1^q+a_2^q\ldots+a_n^q}.$$

The geometrical mean

$$=\{a_1^{(p-q)a_1^q}.a_2^{(p-q)a_2^q}\ldots a_n^{(p-q)a_n^q}\}^{\frac{1}{a_1^q+\ldots+a_n^q}}$$

$$=\{(a_1^q)^{a_1^q}.(a_2^q)^{a_2^q}\ldots(a_n^q)^{a_n^q}\}^{\frac{p-q}{q\{a_1^q+a_2^q+\ldots+a_n^q\}}}$$

$$>\left\{\left(\frac{a_1^q+a_2^q+\ldots+a_n^q}{n}\right)^{a_1^q+\ldots+a_n^q}\right\}^{\frac{p-q}{q\{a_1^q+\ldots+a_n^q\}}}$$

$$>\left\{\frac{a_1^q+a_2^q\ldots+a_n^q}{n}\right\}^{\frac{p-q}{q}};$$

therefore $$\frac{a_1^p+a_2^p\ldots+a_n^p}{a_1^q+a_2^q\ldots+a_n^q}>\left\{\frac{a_1^q+a_2^q\ldots+a_n^q}{n}\right\}^{\frac{p-q}{q}},$$

$$\frac{\{a_1^p+a_2^p+\ldots+a_n^p\}^q}{\{a_1^q+a_2^q+\ldots+a_n^q\}^q}>\left\{\frac{a_1^q+a_2^q\ldots+a_n^q}{n}\right\}^{p-q};$$

therefore $$\frac{\{a_1^p+a_2^p\ldots+a_n^p\}^q}{\{a_1^q+a_2^q\ldots+a_n^q\}^p}>\frac{1}{n^{p-q}}.$$

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A GRAPHIC REPRESENTATION OF THE THEOREMS OF STURM AND FOURIER.

By *H. W. Lloyd Tanner.*

FOR some time past I have made use of the following method of illustrating Sturm's theorem relative to the real roots of algebraical equations, chiefly in the hope of giving a clear conception of the way in which the distribution of signs varies from root to root. As the process has proved convenient, and appears capable of extended application, a note upon it may be of interest.

Sturm's Theorem (figs. *A*, *B*, *C*). The values of x, the unknown quantity in a proposed equation, are distributed along a horizontal line, the axis of x. Above this line the plane of the diagram is divided into strips by lines parallel to the axis of x. To each of Sturm's functions one strip is assigned, consecutive functions having adjacent strips. The function whose roots are to be analysed takes the top strip, and the other functions are placed in order below it. The diagram is bounded on the left and right by vertical lines corresponding to the extreme values of x considered.

The strip belonging to each function is divided into segments by vertical lines drawn vertically over those points in the axis of x, which denote the real roots of the function. In each of these segments the sign of the function is unique; and in adjacent segments of one strip the signs are opposite, except, it may be, when the corresponding function has equal roots, a case provisionally excluded. When every strip has been marked in this way the lines, both horizontal and vertical, which separate segments of opposite sign, are distinguished from the others by being made thicker. The thick lines form a number of continuous clues which are called change-lines. Since the positive regions of the diagram are separated from the negative regions by these change-lines, each of these lines must either have its two ends on the boundary of the diagram or form a closed circuit in the diagram. It will appear that only the former alternative occurs.

No change-line can meet the axis or bottom of the diagram, for the lowest strip is of one sign throughout (the case of equal roots being excluded). In the body of the diagram the shape of the change-lines is determined by the property that $f_{s+1}(x)$ and $f_{s-1}(x)$ are of opposite signs when $f_s(x)$ vanishes. This requires that the change-lines should consist of horizontal lines connected by junctions of the

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shapes ⌜, ⌜; while the turns ⌝, ⌞ do not occur. Hence we infer that no change-line forms a complete circuit, and that no change-line can have both ends on the same side boundary, for in either case one of the excluded turns must be made. A change-line passing into the highest strip must make the turn ⌟. This is the graphic equivalent of the facts that f' is +, when f changes from − to +, and − when f changes from + to −. Since the turn ⌞ is excluded, no change-line can have one end at the top and the other at the right boundary. Nor can any change-line have both ends at the top. In fine, it appears that every change-line has one end at the left boundary and its other end either at the top (denoting a real root of fx) or at the right boundary. And this is, in effect, Sturm's theorem, for the number of change-lines that meet a vertical line is the number of changes of sign in the set of Sturm functions for the corresponding value of x.

Fourier's Theorem (figs. *A*, *D*). From the diagram which answers to Fourier's theorem the only turn excluded is ⌞, as at the top of the Sturm-diagram, for each function is the derivative of the one above it. Since this turn is not found in the diagram there can be no circuit in the diagram, and no change-line can have one end at the top and the other end either at the top or at the right boundary. But a change-line may begin and end in the left boundary, since, to do this, it makes only the permissible turns ⌟, ⌜, ⌝, ⌟. Thus there are three kinds of lines:—(1) lines proceeding from the left boundary to the top, and so corresponding to real roots; (2) lines proceeding from the left to the right boundary (only possible when some of the functions have real roots beyond the limits of the diagram; (3) lines which turn downward or towards the left at some point of their course, after which they continue to move downwards and leftwards and end on the left boundary (fig. *D*). Every line of this kind, since it has both ends at the left, may be regarded as consisting of two change-lines which cancel each other.

In a complete diagram (*i.e.* one which includes all the real roots of fx and its derivatives) all the lines are change-lines at the left and none at the right; so that the change-lines indicate *roots*, not merely *real roots*. The change-lines that cross a vertical ($x=a$) indicate either imaginary roots or real roots greater than a. And this is in substance Fourier's theorem.

An obvious corollary is that every ⌝ in a Fourier diagram anwers to a pair of imaginary roots, and conversely. Since

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each retort line has one and only one ⌝. The analytical equivalent of a ⌝ is that $fx.f'x$ changes from − to +, while fx does not change sign; but this, as it stands, is of no practical value, since it postulates the examination of the sign of a function throughout an interval. Another way of expressing the matter is by saying that the Sturm rule is broken; viz. when one of the functions vanishes the adjacent functions have the same sign.

Of the figures illustrating this note the first (fig. *A*) refers to a quartic with all its roots real, and may be either a Sturm- or a Fourier-diagram. The others represent a quartic with a pair of imaginary roots, figs. *B* and *C* are Sturm-diagrams, fig. *D* a Fourier-diagram.

The modifications caused by the presence of equal roots follow the lines of text-books. In the case of a Fourier-diagram it is convenient to use a penultimate form.

Number of Sturm-diagrams. We proceed to determine the number of complete Sturm-diagrams belonging to a quantic of given degree. A necessary preliminary is a statement of what are to be counted as distinct diagrams.

In fig. *B*, f_2 has two real roots which are not separated by roots of either of the adjacent functions; say these are 'sub-equal' roots. If these approach each other there is no change in the configuration until they become equal. When this happens the change-line ceases to cross the f_2-strip, and the diagram assumes the same appearance as if f_2 had two imaginary roots. We consider two diagrams which differ only by reason of the presence of sub-equal roots to be equivalent (for instance, figs. *B*, *C* are equivalent), and we shall only count the diagrams from which all sub-equal roots have been removed. This reduction does not in any way affect the disposition of the lines at the boundary. The reduced diagrams are distinguished by the property that the roots of every strip are separated by those of the adjacent strips. A reduced diagram is uniquely determined by the arrangement of the change-lines at the left boundary. For each horizontal line is a change-line at one end only; the number of change-lines at the top is thus fixed. But, the two ends of every change line being given, it is easy to see that in only one way can we pass from one end to the other without crossing the same strip twice; namely, we must continually ascend or continually descend. The question is therefore reduced to finding the number of arrangements of the change-lines at the left-boundary. The conditions limiting

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the arrangement are (1) that the line between f and f' is always a change line, and (2) that not less than half the horizontal lines are change-lines. Hence it is found without difficulty that the number of reduced Sturm-diagrams for n-ic is $2^{n-2}+2^{m-1}1.3.5...(2m-1)/(1.2.3...m)$, where $n=2m$ or $2m+1$. Calling this number N, we have, when

$$n = 2,\ 3,\ 4,\ 5,\ 6,\ 7,\ 8,\ \ldots,$$
$$N = 2,\ 3,\ 7,\ 11,\ 26,\ 42,\ 99,\ \ldots.$$

If we counted as distinct those diagrams which differ by reason of the presence and arrangement of sub-equal roots, these numbers would be largely increased. For instance, the two diagrams in figs. B, C belong to a set of nine, which are equivalent to one reduced diagram. I do not see how to get a general expression for this larger number. As some explanation of the great number of the diagrams, it may be remarked that they are based on the properties of Sturm's functions, and not on any particular method of determining such functions, so that the diagrams include every kind of Sturm's functions.

Resultants and discriminants. Another feature of the diagram is to be noted. Each horizontal line represents the resultant of the two functions it separates, and the sign of this resultant is read off at once. Suppose the two functions are g, h, and that a root of the latter is b. If b is in a negative segment of g this gives $g(b)$ negative, so that b contributes a negative factor to the resultant. Hence we need only count the roots of h which occur in negative segments of g. But a considerable simplification can be introduced, for it is clear that any pair of roots of h which occurs in a segment of g may be removed without affecting the sign of the result; that is to say, a reduced diagram is sufficient. But in the reduced diagram the roots of each function are separated by the roots of the other. Hence the sign of the resultant is found by counting the number of negative segments of g into which roots of h enter; that is to say, all the negative segments if g is below h; or all except the extreme segments if g is above h. It is easy to verify that the sign of the resultant is the same whether we put the roots of g into h or the roots of h into g; and this is right, because one of the functions must be of even degree.

In particular, considering the part of the diagram containing f and f', we find that the sign of the discriminant of f is + or −, according as there are an even or odd number of negative mean segments in the top strip of the diagram.

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If an equation of the n^{th} degree has i pairs of imaginary roots and r real roots, the discriminant is positive when $r, = n-2i, = 4m$ or $4m+1$ (for then there are $2m$ mean negative segments); and it is negative when $r, = n-2i, = 4m+2$ or $4m+3$, for then there are $2m+1$ mean negative segments. And conversely, if the discriminant of an n-ic is positive, there are $n-4m$ or $n-4m-1$ (whichever is even) imaginary roots; if the discriminant is negative, there are $n-4m-2$ or $n-4m-3$ imaginary roots where m is an undetermined integer.

Fig. A.

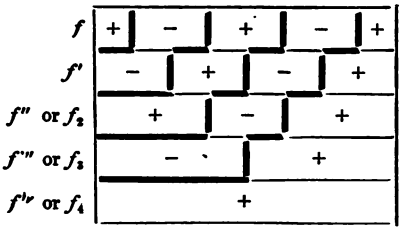


Fig. B.

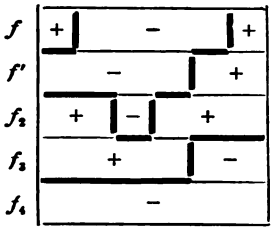


Fig. C.

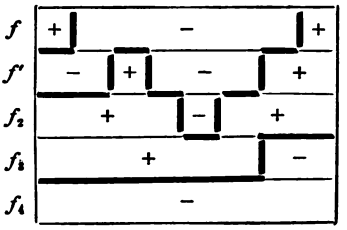
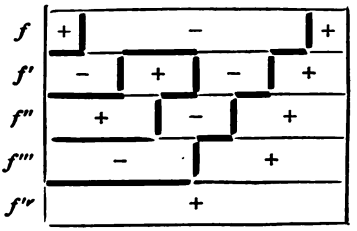


Fig. D.



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NOTE ON THE RELATION BETWEEN THE DISTANCE OF FIVE POINTS IN SPACE.

By Prof. *Cayley*.

In Lagrange's paper "Solutions Analytiques de quelques problèmes sur les pyramides triangulaires" (*Berlin Memoirs* 1773, *Œuvres* t. iii., see p. 677) there is contained a formula for the relation between the distances from each other of five points in space; viz. this is

$$4\Delta^2 f = \alpha(a+f-g)^2 + \alpha'(a'+f-g')^2 + \alpha''(a''+f-g'')^2$$
$$+ 2\beta(a'+f-g')(a''+f-g'') + 2\beta'(a+f-g)(a''+f-g'')$$
$$+ 2\beta''(a+f-g)(a'+f-g'),$$

or in a slightly altered notation, say

$$\Pi = -4\Delta^2 f + (\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3)(a_1+f-g_1, a_2+f-g_2, a_3+f-g_3)^2 = 0,$$

where if the points are called 1, 2, 3, 4, 5, then

c_1, c_2, c_3	are the squared distances		23, 31, 12,
a_1, a_2, a_3		„	41, 42, 43,
g_1, g_2, g_3		„	51, 52, 53,
and f	is the square distance		45.

The values of α_1, α_2, α_3, β_1, β_2, β_3 in terms of these squared distances are

$$\alpha_1 = -(a_3-a_2)^2 + 2c_1(a_3+a_2) - c_1^2,$$
$$\alpha_2 = -(a_3-a_1)^2 + 2c_2(a_3+a_1) - c_2^2,$$
$$\alpha_3 = -(a_1-a_2)^2 + 2c_3(a_1+a_2) - c_3^2,$$
$$\beta_1 = -(a_3-a_1)(a_1-a_2) + 2c_1a_1 - c_2(a_1+a_2) - c_3(a_3+a_1) + c_2c_3,$$
$$\beta_2 = -(a_1-a_2)(a_2-a_3) + 2c_2a_2 - c_3(a_2+a_3) - c_1(a_1+a_2) + c_3c_1,$$
$$\beta_3 = -(a_2-a_3)(a_3-a_1) + 2c_3a - c_1(a_3+a_1) - c_2(a_2+a_3) + c_1c_2;$$

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and after some reductions the value of $4\Delta^2$ in terms of these squared distances is found to be

$$\begin{aligned}4\Delta^2 = \; & c_1(a_3-a_1)(a_1-a_2)+a_1(-c_1^2+c_1c_2+c_1c_3)-c_1c_2c_3\\ & +c_2(a_1-a_2)(a_2-a_3)+a_2(-c_2^2+c_2c_3+c_2c_1)\\ & +c_3(a_2-a_3)(a_3-a_1)+a_3(-c_3^2+c_3c_1+c_3c_2),\end{aligned}$$

which is in fact

$$8\Delta^2 = \begin{vmatrix} . & 1, & 1, & 1, & 1 \\ 1, & . & c_3 & c_2 & a_1 \\ 1, & c_3 & . & c_1 & a_2 \\ 1, & c_2 & c_1 & . & a_3 \\ 1, & a_1, & a_2, & a_3, & . \end{vmatrix}$$

(where observe that each term of the determinant contains the factor 2).

By the formula in my paper "*On a theorem in the Geometry of Position*," *Camb. Math. Jour.*, t. ii. (1841), pp. 267—271, (introducing into it the present notation), the relation between the distances of the five points is given in the form

$$\Omega = \begin{vmatrix} . & 1, & 1, & 1, & 1, & 1 \\ 1, & . & c_3, & c_2, & a_1, & g_1 \\ 1, & c_3, & . & c_1, & a_2, & g_2 \\ 1, & c_2, & c_1, & . & a_3, & g_3 \\ 1, & a_1, & a_2, & a_3, & . & f \\ 1, & g_1, & g_2, & g_3, & f & . \end{vmatrix} = 0.$$

The equations $\Pi = 0$, $\Omega = 0$ should therefore agree with each other; we have in Ω the term

$$-f^2 \begin{vmatrix} 1, & c_3, & . & c_1 \\ 1, & c_2, & c_1, & . \\ . & 1, & 1, & 1 \\ 1, & . & c_3, & c_2 \end{vmatrix},$$

which is

$$= -f^2(c_1^2+c_2^2+c_3^2-2c_2c_3-2c_3c_1-2c_1c_2);$$

and similarly in Π we have the term

$$f^2(\alpha_1+\alpha_2+\alpha_3+2\beta_1+2\beta_2+2\beta_3),$$

which is easily shown to be

$$= -f^2(c_1^2+c_2^2+c_3^2-2c_2c_3-2c_3c_1-2c_1c_2);$$

and it thus appears that we have identically $\Pi = \Omega$.

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It is to be remarked that in Lagrange's form the points 4 and 5 are regarded as determined each of them by means of its squared distances from the vertices of the triangle 123, and that the formula gives (by a quadratic equation) the squared distance 45; but that nevertheless the two points 4 and 5 do not present themselves symmetrically in the formula; in fact, Δ^2 and the coefficients $(\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3)$ of the formula relate all of them to the tetrahedron 4123; as noticed in the memoir $\Delta = 6 \times$ volume of tetrahedron; $\sqrt{(\alpha_1)}$, $\sqrt{(\alpha_2)}$, $\sqrt{(\alpha_3)}$ are the doubled areas of the faces 423, 431, 413 respectively (and the doubled area of the face 123 is $= \sqrt{(\alpha_1 + \alpha_2 + \alpha_3 + 2\beta_1 + 2\beta_2 + 2\beta_3)}$; it may be added that $\beta_1 \div \sqrt{(\alpha_2\alpha_3)}$, $\beta_2 \div \sqrt{(\alpha_3\alpha_1)}$, $\beta_3 \div \sqrt{(\alpha_1\alpha_2)}$ are equal to the cosines of the dihedral angles at the edges 41, 42, 43 respectively.

NOTE ON MATRICES IN INVOLUTION.

By *Arthur Buchheim, M.A.*

If we take two matrices a, b of order n, the n^2 products of the form $a^i b^k (i, k = 0 \ldots n-1)$ will in general be asyzygetic. If they are syzygetic, a, b are said to be in involution. This definition is Prof. Sylvester's, and he has considered the theory in one of the Johns Hopkins Circulars. His main result is that the syzygy must be of the form

$$(a - \lambda')(a - \lambda'') \ldots (b - \mu')(b - \mu'') \ldots = 0,$$

where $\lambda', \lambda'', \ldots$, are $n-1$ of the latent roots of a, and $\mu', \mu'', \ldots$, are $n-1$ of the latent roots of b.

Now if we write the left-hand side of this equation in the form $f(a).F(b)$ we see that $f(a)$ and $F(b)$ are both of nullity $n-1$, so that we have two matrices of nullity $n-1$, whose product is of nullity n. Now I have proved in a paper on the theory of matrices in the *Proc. L. M. S.*, and in a note in the *Phil. Mag.** that Prof. Sylvester's theorem respecting the nullity of a product may be enunciated in the following more precise form; If we have two matrices a, b, and b is of nullity r, then if the latent space of b cuts the null space of a in an s-point, the nullity of ab is $r + s$.†

* This note was entitled a "Proof of Prof. Sylvester's 'Third Law of Motion,'" and accordingly the editors of the *Fortschritte der Mathematik* have included it among the papers on dynamics in the volume for 1884.

† I do not think that I have brought out the obviousness of the theorem in either proof. If a point is to be destroyed by ab it must either be destroyed by b or else b must prepare it for destruction by a, but b brings all points into its latent space, a only destroys points in its null space, and the theorem follows at once.

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If we apply this to the product $f(a).F(b)$ we see that, since the nullity of the product exceeds that of the second factor by unity, the latent space of $F(b)$ must cut the null space of $f(a)$ in a point; but the latent space of $F(b)$ is the latent point of b corresponding to the latent root excluded from the set μ', μ'', &c., and the null space of $f(a)$ is the $(n-1)$ point determined by the latent points corresponding to μ', μ'', &c. Prof. Sylvester's theorem can therefore be enunciated as follows: The necessary (obviously sufficient) condition that two matrices may be in involution is that a latent point of the one must be in the $(n-1)$ point determined by $(n-1)$ of the latent points of the other.

The object of the present note is to prove this theorem directly.

If we have a syzygy among the products $a^i b^k$ it is obvious that we can write it in the form

$$F = a^{n-1} f_1(b) + a^{n-2} f_2 b + \ldots = 0,$$

where f_1, f_2, &c., are scalar functions of order $n-1$ at most.

Now let η be any latent point of b, μ the corresponding latent root, let $\lambda_1 \ldots \lambda_n$ be the latent roots, $e_1 \ldots e_n$ the corresponding latent points of a. Lastly, let

$$\eta = x_1 e_1 + x_2 e_2 + \ldots + x_n e_n,$$

then I shall show that one at least of the x's must vanish.

We have

$$\begin{aligned}
0 &= F\eta \\
&= f_1(\mu).a^{n-1}\eta + f_2(\mu)\,a^{n-2}\eta + \ldots \\
&= f_1(\mu)(\lambda_1^{n-1} x_1 e_1 + \lambda_2^{n-1} x_2 e_2 + \ldots) \\
&\quad + f_2(\mu)(\lambda_1^{n-2} x_1 e_1 + \lambda_2^{n-2} x_2 e_2 + \ldots) \\
&\quad + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
&= e_1 . x_1 [\lambda_1^{n-1} f_1(\mu) + \lambda_1^{n-2} f_2(\mu) - \ldots] \\
&\quad + e_2 . x_2 [\lambda_2^{n} f_1(\mu) + \lambda_2^{(n-2)} f_2(\mu) + \ldots] \\
&\quad + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots .
\end{aligned}$$

But the points e_1, &c. are asyzygetic, and we must therefore have

$$\begin{aligned}
x_1 \phi(\lambda_1) &= 0, \\
x_2 \phi(\lambda_2) &= 0, \\
&\ldots\ldots\ldots\ldots,
\end{aligned}$$

where $\phi(\lambda_1)$, $\phi(\lambda_2)$, &c. denote the quantities in the square brackets.

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Therefore, if none of the x's vanish, the $(n-1)$-thic function $\phi(x)$ must have n zeros, which is impossible unless $f_1(\mu)$, $f_2(\mu)$, &c. all vanish; but this can always be prevented since f_1, f_2 are at most of order $n-1$, and cannot all vanish for all the latent roots of b.

Therefore one at least of the x's must vanish, that is, η must be in the $(n-1)$ point determined by $n-1$ of the latent points of a, which was to be proved.

This being so, it follows at once that we have a relation of the form $f(a).F(b)=0$, where f, F are $(n-1)$-thic factors of the latent functions of a, b respectively.

The above proof assumes that a, b are not derogatory. It is not hard to extend the theorem and the proof to the case of derogatory matrices, but the theorem becomes rather cumbrous and the method of proof remains the same in all essential respects. The theorem admits of very obvious generalisations.

The Grammar School, Manchester,
Sept. 23, 1887.

ON HERMITE'S H-PRODUCT THEOREM.

By Prof. *Cayley*.

I GIVE this name to a theorem relating to the product of an even number of Eta-functions, established by M. Hermite in his "Note sur le Calcul differential et le Calcul integral" forming an appendix to the sixth edition of Lacroix's differential and integral Calculus, and separately printed 8°, Paris, 1862: it is the theorem stated p. 65, in the form

$$\phi(x)=F(z^2)+\frac{dz}{dx}zF_1(z^2),$$

where

$$\phi(x)=\frac{AH(x-\alpha_1)\,H(x-\alpha_2)\ldots H(x-\alpha_{2n})}{\Theta^{2n}(x)},$$

where $\alpha_1+\alpha_2\ldots+\alpha_{2n}=0$, and $z=\mathrm{sn}\,x$, $\mathrm{cn}\,x$ or $\mathrm{dn}\,x$ at pleasure; $F(z^2)$, $F_1(z^2)$ denote rational and integral functions of z^2 of the degrees n and $n-2$ respectively; A is a constant, which we may if we please so determine that in $F(z^2)$ the coefficient of the highest power z^{2n} shall be $=1$.

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If, for shortness, we write s, c, d for $\operatorname{sn} x, \operatorname{cn} x, \operatorname{dn} x$ respectively; and to fix the ideas assume $z = \operatorname{sn} x$, $= s$, then the theorem is

$$\frac{AH(x-\alpha_1)\,H(x-\alpha_2)\ldots H(x-\alpha_{2n})}{\Theta^{2n}(x)} = F(s^2) + scdF_1(s^2);$$

viz., the theorem is that the product of the $2n$ H-functions ($\alpha_1 + \alpha_2 \ldots + \alpha_{2n} = 0$ as above) divided by $\Theta^{2n}(x)$ is a function of the elliptic functions sn, cn, dn, of the form in question.

Hermite uses the theorem for the demonstration of Abel's theorem, as applied to the elliptic functions; or as I would rather express it, he uses the theorem for the determination of the sn, cn, and dn of $\alpha_1 + \alpha_2 \ldots + \alpha_{2n-1}$.

To show how this is, observe that $F(s^2)$, quà rational and integral function of s^2 of the degree n, with its first coefficient $= 1$, contains n arbitrary coefficients; and $F_1(s^2)$, quà rational and integral function of s^2 of the degree $n-2$ contains $n-1$ arbitrary coefficients: hence $F(s^2) + scdF_1(s^2)$ contains $2n-1$ arbitrary coefficients; and considering $\alpha_1, \alpha_2 \ldots \alpha_{2n-1}$ as given, the function in question must vanish for each of the values $x = \alpha_1, \alpha_2, \ldots \alpha_{2n-1}$; and we have therefore $2n-1$ equations for obtaining the $2n-1$ coefficients, which are thus completely determined: in particular the constant term, say L, of $F_1(s^2)$ is a given function of $\alpha_1, \alpha_2, \ldots \alpha_{2n-1}$, that is of the sn, cn, and dn of these quantities; and the theorem shows that the function thus determined vanishes also for $x = \alpha_{2n}$, that is $= -(\alpha_1 + \alpha_2 \ldots + \alpha_{2n-1})$.

Now in the formula writing $-x$ for x, and recollecting that H is an odd function, Θ an even function, we find

$$\frac{AH(x+\alpha_1)\,H(x+\alpha_2)\ldots H(x+\alpha_{2n})}{\Theta^{2n}(x)} = F(s^2) - scdF_1(s^2);$$

and multiplying together the two sides of these equations respectively,

$$A^2 \frac{H(x-\alpha_1)\,H(x+\alpha_1)}{\Theta^2(x)} \ldots \frac{H(x-\alpha_{2n})\,H(x+\alpha_{2n})}{\Theta^2(x)}$$
$$= \{F(s^2)\}^2 - s^2c^2d^2\{F_1(s^2)\}^2,$$

where the right-hand side is a rational and integral function of s^2 of the degree $2n$, and the coefficient of the highest term s^{4n} is $= 1$; in fact this term arises only from the square of $F(s^2)$, which has its highest term $= s^{2n}$.

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Now $\dfrac{H(x-\alpha_1)\,H(x+\alpha_1)}{\Theta^2(x)}$ is a mere constant multiple of $\mathrm{sn}^2x-\mathrm{sn}^2\alpha_1$, or say of $s^2-\mathrm{sn}^2\alpha_1$; (this well-known theorem is in fact the particular case $n=2$ of Hermite's theorem); and similarly for the other terms: we must clearly have A^2 into the product of the factors thus introduced, $=1$; and thus the theorem becomes

$$(s^2-\mathrm{sn}^2\alpha_1)(s^2-\mathrm{sn}^2\alpha_2)\ldots(s^2-\mathrm{sn}^2\alpha_{2n})=\{F(s^2)\}^2-s^2c^2d^2\{F_1(s^2)\}^2.$$

And putting herein $s=0$, and writing as before L for the constant term of $F(s^2)$, we have

$$\mathrm{sn}^2\alpha_1\,\mathrm{sn}^2\alpha_2\ldots\mathrm{sn}^2\alpha_{2n}=L^2,$$

or, the sign $\pm$ being properly determined, say

$$\mathrm{sn}\,\alpha_1\,\mathrm{sn}\,\alpha_2\ldots\mathrm{sn}\,\alpha_{2n}=\pm L,$$

where by what precedes L is a given function of the sn, cn, and dn of $\alpha_1, \alpha_2\ldots\alpha_{2n-1}$. Hence we have $\mathrm{sn}\,\alpha_{2n}$, that is, $-\mathrm{sn}(\alpha_1+\alpha_2\ldots+\alpha_{2n-1})$ as a given function of the sn, cn, and dn of $\alpha_1, \alpha_2, \ldots \alpha_{2n-1}$.

Similarly writing $z=\mathrm{cn}\,x$, $=c$, and $z=\mathrm{dn}\,x$, $=d$, we have $\mathrm{cn}(\alpha_1+\alpha_2\ldots+\alpha_{2n-1})$ and $\mathrm{dn}(\alpha_1+\alpha_2\ldots+\alpha_{2n-1})$ each of them as a given function of the sn, cn, and dn of $\alpha_1, \alpha_2, \ldots \alpha_{2n-1}$.

It is hardly necessary to remark that $F(z^2)+z\dfrac{dz}{dx}F_1(z^2)$ is a function of the same form whether we have $z=s$, c or d; in fact the functions F and F_1 are rational in s^2, c^2, or d^2, and we have $z\dfrac{dz}{dx}=scd$, $-scd$, and $-k^2scd$ for the three values respectively.

The number of terms $\alpha_1, \alpha_2, \ldots \alpha_{2n-1}$ has been odd, but by taking one of them $=0$, the formulæ give the values of the sn, cn, and dn for the sum of an even number of terms.

It has been seen that Hermite's H-product theorem gives, say Abel's theorem, in the form

$$\Pi(s^2-\mathrm{sn}^2\alpha)=\{F(s^2)\}^2-s^2c^2d^2\{F_1(s^2)\}^2,$$

each side of this relation being the product of two factors, viz., for the left-hand side the factors are

$$A\Pi\frac{H(x-\alpha)}{\Theta(x)},\quad A\Pi\frac{H(x+\alpha)}{\Theta(x)},$$

and for the right-hand side they are their rational functions of s^2,

$$F(s^2)+scdF_1(s^2),\quad F(s^2)-scdF_1(s^2);$$

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these factors are by Hermite's theorem equal each to each; viz., this is the relation in which Hermite's stands to Abel's theorem.

The H-product theorem is given as one out of a group of four theorems; the other three may be called the H-product, H_1-product and Θ_1-product, odd theorems respectively,

$$\{\Theta_1(x) = \Theta(x+K),\ H_1(x) = H_1(x+K)\},$$

viz., these are

$$\begin{cases} \dfrac{A_1 H(x-\alpha_1) H(x-\alpha_2)\ldots H(x-\alpha_{2n+1})}{\Theta^{2n+1}(x)} = s F(s^2) + cd\phi(s^2), \\ \dfrac{A_2 H_1(x-\alpha_1) H_1(x-\alpha_2)\ldots H_1(x-\alpha_{2n+1})}{\Theta^{2n+1}(x)} = c F(c^2) - sd\phi(c^2), \\ \dfrac{A_3 \Theta_1(x-\alpha_1) \Theta_1(x-\alpha_2)\ldots \Theta_1(x-\alpha_{2n+1})}{\Theta^{2n+1}(x)} = d F(d^2) - k^2 sc\phi(c^2), \end{cases}$$

where F, ϕ are rational and integral functions of the degrees n and $n-1$, having their proper values in the three equations respectively, and in each case $\alpha_1 + \alpha_2 \ldots + \alpha_{2n+1} = 0$.

It was seen above that for $n=1$, the H-product theorem became

$$\frac{AH(x-\alpha) H(x+\alpha)}{\Theta^2(x)} = \mathrm{sn}^2 x - \mathrm{sn}^2 \alpha,$$

which is the most simple case; for the odd theorems the most simple case is $n=0$, viz., we then have

$$\frac{A_1 H(x)}{\Theta(x)} = \mathrm{sn}\, x,\quad \frac{A_2 H_1(x)}{\Theta(x)} = \mathrm{cn}\, x,\quad \frac{A_3 \Theta_1(x)}{\Theta(x)} = \mathrm{dn}\, x;$$

to complete the formulæ observe that the values of the constants are

$$A = \frac{2k'K}{k\pi\Theta^2(\alpha)},\ A_1 = \frac{1}{\sqrt{(k)}},\ A_2 = \sqrt{\left(\frac{k'}{k}\right)},\ A_3 = \sqrt{(k')}.$$

The three theorems may be used, in like manner with the H-product theorem, to give the values of the sum $\alpha_1 + \alpha_2 \ldots + \alpha_{2n}$ of the sn, cn, and dn respectively.

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ON THE EQUATIONS OF MOTION OF A SOUND WAVE IN A COMPRESSIBLE FLUID WHICH IS RENDERED HETEROGENEOUS BY A CONSTANT ACCELERATING FORCE IN A GIVEN DIRECTION.

By *R. Holmes, B.A.*

We shall suppose that the temperature throughout the fluid is constant, and that the axis of z is taken parallel to the direction of the constant accelerating force g. Let p, ρ be the equilibrium pressure and density at x, y, z, then we have

$$\frac{1}{\rho}\frac{dp}{dz} = g \text{ and } p = \kappa\rho.$$

Now suppose that $p+p'$, $\rho+\rho'$ are the disturbed pressures and densities at that point when there is wave motion, and that u, v, w are the velocities parallel to the axes; the hydrodynamical equations are

$$\frac{1}{\rho+\rho'}\frac{dp'}{dx} = -u\frac{du}{dx} - v\frac{du}{dy} - w\frac{du}{dz} - \frac{du}{dt},$$

$$\frac{1}{\rho+\rho'}\frac{dp'}{dy} = -u\frac{dv}{dx} - v\frac{dv}{dy} - w\frac{dv}{dz} - \frac{dv}{dt},$$

$$\frac{1}{\rho+\rho'}\frac{d(p+p')}{dz} = g - u\frac{dw}{dx} - v\frac{dw}{dy} - w\frac{dw}{dz} - \frac{dw}{dt},$$

together with the so-called equation of continuity

$$\frac{d(\rho')}{dt} + \frac{d(\rho+\rho')u}{dx} + \frac{d(\rho+\rho')v}{dy} + \frac{d(\rho+\rho')w}{dz} = 0.$$

Hence to the first order of p', ρ', u, v, w, we have, putting $\rho' = s\rho$,

$$\left.\begin{aligned}
&\frac{1}{\rho}\frac{dp'}{dx} = -\frac{du}{dt} \quad \text{.....(i)},\\
&\frac{1}{\rho}\frac{dp'}{dy} = -\frac{dv}{dt} \quad \text{.....(ii)},\\
&\frac{1}{\rho}\frac{dp'}{dz} - \frac{s}{\rho}\frac{dp}{dz} = -\frac{dw}{dt} \quad \text{.....(iii)},\\
&\frac{d\rho s}{dt} + \frac{d\rho u}{dx} + \frac{d\rho v}{dy} + \frac{d\rho w}{dz} = 0 \quad \text{.....(iv)}.
\end{aligned}\right\}$$

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Now, whenever we have compression or rarefaction of air due to a wave of sound, on the supposition that there is no ingress or egress of heat, we have the relation that the change of pressure is γ times as great as it would have been had there been no change of temperature. Hence, we have

$$\frac{1}{P}\frac{\partial P}{\partial t} = \frac{\gamma}{R}\frac{\partial R}{\partial t}.$$

But $$\frac{\partial}{\partial t} = \frac{d}{dt} + u\frac{d}{dx} + v\frac{d}{dy} + w\frac{d}{dz}.$$

Hence $$\frac{1}{p}\left(\frac{dp'}{dt} + w\frac{dp}{dz}\right) = \frac{\gamma}{\rho}\left(\rho\frac{ds}{dt} + w\frac{d\rho}{dz}\right) \ldots\ldots\ldots (v).$$

Now equation (iv) may be written

$$\frac{ds}{dt} + \frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz} + \frac{1}{\rho}w\frac{d\rho}{dz} = 0 \ldots\ldots\ldots (vi),$$

so that equation (v) becomes

$$\frac{1}{\rho}\left(\frac{dp'}{dt} + w\frac{dp}{dz}\right) + \gamma\left(\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}\right) = 0 \ldots\ldots (vii).$$

Hence, differentiating (i), (ii), (iii) with regard to t, and eliminating $\frac{ds}{dt}$, $\frac{dp'}{dt}$ by means of (vi) and (vii), we obtain

$$\left.\begin{aligned}
&\frac{d}{dx}\left(\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz} + \frac{1}{\gamma}\frac{d\log p}{dz}w\right) = \frac{1}{k\gamma}\frac{d^2u}{dt^2}\\
&\frac{d}{dy}\left(\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz} + \frac{1}{\gamma}\frac{d\log p}{dz}w\right) = \frac{1}{k\gamma}\frac{d^2v}{dt^2}\\
&\frac{d}{dz}\left(\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz} + \frac{1}{\gamma}\frac{d\log p}{dz}w\right)\\
&\qquad + \frac{\gamma-1}{\gamma}\frac{d\log p}{dz}\left(\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}\right) = \frac{1}{k\gamma}\frac{d^2w}{dt^2}
\end{aligned}\right\} \ldots (A).$$

Assuming that u, v, w vary as e^{int}, and putting $a = \frac{g}{k\gamma}$, $m = \frac{n^2}{g}$, the equations (A) become

$$\left.\begin{aligned}
&\frac{d}{dx}\left(\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{d} + aw\right) + amu = 0,\\
&\frac{d}{dy}\left(\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz} + aw\right) + amv = 0,\\
&\frac{d}{dz}\left(\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz} + aw\right) + a(\gamma-1)\left(\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}\right) + amw = 0.
\end{aligned}\right\}$$

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Now let us limit our investigation to the case in which the motion has a velocity potential, so that we may take

$$u=\frac{d\phi}{dx},\quad v=\frac{d\phi}{dy},\quad w=\frac{d\phi}{dz},$$

where ϕ is the velocity potential, and our equations become

$$\left.\begin{aligned}
&\frac{d}{dx}\left(\nabla^2\phi+a\frac{d\phi}{dz}+am\phi\right)=0,\\
&\frac{d}{dy}\left(\nabla^2\phi+a\frac{d\phi}{dz}+am\phi\right)=0,\\
&\frac{d}{dz}\left(\nabla^2\phi+a\frac{d\phi}{dz}+am\phi\right)+a(\gamma-1)\nabla^2\phi=0,
\end{aligned}\right\}$$

when

$$\nabla^2=\frac{d^2}{dx^2}+\frac{d^2}{dy^2}+\frac{d^2}{dz^2}.$$

A complete solution of these equations when there is no source at the origin may be written

$$\phi=\sum_{q=0}^{q=\infty}A_q e^{-mt}J_q\{m(x^2+y^2)\}\cos(q\theta+\theta_q)+Be^{\alpha z}\cos\beta z+De^{\alpha z}\sin\beta z,$$

where J_q is the Bessel's function of order q; θ is the longitudinal angle; $\alpha\pm\beta\sqrt{(-1)}$ are the roots of the equation $h^2+\gamma ah+am=0$; assuming for the present that these roots are imaginary.

To obtain the solution in the simplest manner, assume $\phi=\psi+\chi$, where χ is a function of z only, and ψ contains xy as well as z. We obtain

$$\left.\begin{aligned}
&\frac{d}{dx}\left(\nabla^2\psi+a\frac{d\psi}{dz}+am\psi\right)=0,\\
&\frac{d}{dy}\left(\nabla^2\psi+a\frac{d\psi}{dy}+am\psi\right)=0,\\
&\frac{d}{dz}\left(\nabla^2\psi+a\frac{d\psi}{dz}+am\psi\right)+a(\gamma-1)\nabla^2\psi\\
&\qquad\qquad+\frac{d}{dz}\left(\frac{d^2\chi}{dz^2}+a\gamma\frac{d\chi}{dz}+am\chi\right)=0.
\end{aligned}\right\}$$

The terms involving χ disappear if we assume

$$\chi=Be^{\alpha z}\cos\beta z+De^{\alpha z}\sin\beta z,$$

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where $\alpha \pm \beta\sqrt{(-1)}$ are roots of $h^2 + a\gamma h + am = 0$; and we obtain

$$\left.\begin{aligned} \frac{d}{dx}\left(\nabla^2\psi + a\frac{d\psi}{dz} + am\psi\right) &= 0, \\ \frac{d}{dy}\left(\nabla^2\psi + a\frac{d\psi}{dz} + am\psi\right) &= 0, \\ \frac{d}{dz}\left(\nabla^2\psi + a\frac{d\psi}{dz} + am\psi\right) + a(\gamma-1)\nabla^2\psi &= 0. \end{aligned}\right\}$$

Now, if we put $\psi = e^{-mz}\psi_1$, we can easily show that ψ_1 must be a function of xy only, and cannot involve z together with x, y. Making the substitutions, our equations become

$$\left.\begin{aligned} \frac{d}{dx}\left[e^{-mz}\left\{\nabla^2\psi_1 + (a-2m)\frac{d\psi_1}{dz} + m^2\psi_1\right\}\right] &= 0, \\ \frac{d}{dy}\left[e^{-mz}\left\{\nabla^2\psi_1 + (a-2m)\frac{d\psi_1}{dz} + m^2\psi_1\right\}\right] &= 0, \\ \frac{d}{dz}\left[e^{-mz}\left\{\nabla^2\psi_1 + (a-2m)\frac{d\psi_1}{dz} + m^2\psi_1\right\}\right] & \\ + a(\gamma-1)e^{-mz}\left(\nabla^2\psi_1 - 2m\frac{d\psi_1}{dz} + m^2\psi_1\right\} &= 0. \end{aligned}\right\}$$

From the first two equations we find

$$\nabla^2\psi_1 + (a-2m)\frac{d\psi_1}{dz} + m^2\psi_1 = e^{mz}f(z),$$

and therefore the third equation gives us

$$\frac{df}{dz} + a(\gamma-1)f(z) = a^2(\gamma-1)\frac{d\psi_1}{dz},$$

shewing that ψ_1 cannot involve xy and z, except in the form

$$F_1(xy) + F_2(z).$$

Hence we obtain the equations

$$\left.\begin{aligned} \frac{d}{dx}\left(\frac{d^2\psi_1}{dx^2} + \frac{d^2\psi_1}{dy^2} + m^2\psi_1\right) &= 0, \\ \frac{d}{dy}\left(\frac{d^2\psi_1}{dx^2} + \frac{d^2\psi_1}{dy^2} + m^2\psi_1\right) &= 0, \\ \frac{d^2\psi_1}{dx^2} + \frac{d^2\psi_1}{dy^2} + m^2\psi_1 &= 0; \end{aligned}\right\}$$

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so that putting $x=\rho\cos\theta$, $y=\rho\sin\theta$, we have

$$\frac{d^2\psi_1}{d\rho^2}+\frac{1}{\rho}\frac{d\psi_1}{d\rho}+\frac{1}{\rho^2}\frac{d^2\psi_1}{d\theta^2}+m^2\psi_1=0,$$

a complete solution of which, neglecting negative powers of ρ is

$$\psi_1=\sum_{q=0}^{q=\infty}A_qJ_q(m\rho)\cos(q\theta+\varepsilon_q).$$

Hence we obtain, for the complete value of the velocity potential ϕ, the expression

$$\sum_{q=0}^{q=\infty}A_qe^{-mz}J_q\{m\sqrt{(x^2+y^2)}\}\cos(q\theta+\varepsilon_q)\cos(nt-\alpha_q)$$
$$+Be^{\alpha z}\cos\beta z\cos(nt-\alpha')+De^{\alpha z}\sin\beta z\cos(nt-\beta'),$$

where $\alpha\pm\beta\sqrt{(-1)}$ are the roots of

$$h^2+\gamma ah+ma=0.$$

Since $$\gamma a=\frac{g}{k},\qquad am=\frac{n^2}{k\gamma};$$

therefore $$\alpha=-\frac{g}{2k},\qquad \beta=\sqrt{\left(\frac{n^2}{k\gamma}-\frac{g^2}{4k^2}\right)},$$

so that $$e^{\alpha z}=e^{-\frac{gz}{2k}}=\sqrt{\left(\frac{p_0}{p}\right)}.$$

And our solution may be written

$$\phi=\sum_{q=0}^{q=\infty}A_qe^{-mz}J_q\{m\sqrt{(x^2+y^2)}\}\cos(q\theta+\varepsilon_q)\cos(nt-\alpha_q)$$
$$+\sqrt{\left(\frac{p_0}{p}\right)}\{B\cos\beta z\cos(nt-\alpha')+D\sin\beta z\cos(nt-\beta')\},$$

where $$m=\frac{n^2}{g},$$

$$\beta=\sqrt{\left(\frac{n^2}{k\gamma}-\frac{g^2}{4k^2}\right)}.$$

It may be easily verified that β is real at any rate for the case of the atmosphere.

This value of ϕ is obtained on the assumption that

$$u\,dx+v\,dy+w\,dz$$

is an exact differential and shows that our investigation is

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limited to one or other of two classes of motion, viz. (1) disturbances propagated in plane waves upwards or downwards, or upwards and downwards; or (2) to disturbances propagated in the two lateral directions but not in the vertical direction.

Of course sound may be propagated obliquely upwards or downwards, but this problem is excluded from the investigation.

The two cases, however, remaining are those possessing most interest, as if we know how the disturbance is propagated vertically and laterally, we should not care to investigate how it is propagated obliquely.

From the value of ϕ for the vertical propagation of plane waves, we see that the velocity of propagation downwards is

$$V_1 = \frac{n}{\sqrt{\left(\frac{n^2}{\kappa\gamma} - \frac{g^2}{4\kappa^2}\right)}} = \frac{\sqrt{(\kappa\gamma)}}{\sqrt{\left(1 - \frac{\gamma g^2}{4\kappa n^2}\right)}},$$

or if V_0 be the velocity of propagation of sound in a homogeneous atmosphere of the same temperature

$$V_1 - V_0 = \frac{g^2\gamma^2}{8n^2 V_0} \text{ approximately,}$$

showing that the effect of gravity is greater on low pitched notes than on high.

NOTE ON CERTAIN DIFFERENCE EQUATIONS WHICH POSSESS AN UNIQUE INTEGRAL.*

By *J. J. Sylvester*, Savilian Professor of Geometry in the University of Oxford.

FOR greater simplicity suppose in what follows that a difference equation is expressed in terms of the arguments

$$u_x, u_{x+1}, \ldots u_{x+i}.$$

I shall call u_{x+i} the highest and u_x the lowest argument respectively, or collectively the extreme or principal arguments, and the degrees in which they enter into the equation the upper and lower or extreme or principal degrees. It is these partial degrees rather than the total degree of the entire equation which determine the essential character of the solution.

* The substance of the following exposition formed the subject of a communication made on Dec. 1st at the second meeting of the Oxford Mathematical Society which has just started into existence.

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If m is the upper degree and $u_0, u_1, \ldots u_{i-1}$ be given it is obvious that for any value of x higher than $(i-1)$, u_x will have m^{x-i+1} values, and consequently in general there will be an infinite number of integrals whether complete or of a given order of deficiency (the deficiency being estimated by the number of relations connecting the initial values $u_0, u_1, \ldots u_{i-1}$); but it may be, and is in some cases, possible to assign an integral which shall have m^{x-i+1} values, and in such case there can exist no other; such an integral may be called an unique or exhaustive one, and the equations which possess such integrals may be termed uni-solutional.

As the simplest example of such, suppose

$$u^m_{x+1} - u_x^n = 0,$$

where m and n are integers.

If we write $$u_x = \alpha^{\left(\frac{n}{m}\right)^x}$$

we have $$u_{x+1} = \left(\alpha^{\left(\frac{n}{m}\right)^x}\right)^{\frac{n}{m}}$$

or $$u^m_{x+1} = u_x^n.$$

Here $u_x = \alpha^{\left(\frac{n}{m}\right)^x}$ is the one and sole complete integral of the equation; for it possesses m^x values so that there can be no other integrals whatever.

Let us now seek to form difference uni-solutional equations of the 2nd order.

To this end let $u_x = C(\alpha^{2^x} - \beta^{2^x})$, where $\alpha\beta = 1$.

Then calling $\alpha^{2^x} = P$ and $\beta^{2^x} = Q$, $PQ = 1$,

$$u_x = C(P - Q),$$
$$u_{x+1} = C(P^2 - Q^2),$$
$$u_{x+2} = C(P^4 - Q^4).$$

Hence $$\frac{u_{x+1}}{u_x} = P + Q, \quad \frac{u_{x+2}}{u_{x+1}} = P^2 + Q^2 = (P+Q)^2 - 2,$$

and $$\frac{u_{x+2}}{u_{x+1}} = \left(\frac{u_{x+1}}{u_x}\right)^2 - 2,$$

Hence the equation

$$u_x^2 u_{x+2} - u^2_{x+1} + 2u_x^2 u_{x+1} = 0$$

has for its complete integral $u_x = C(\alpha^{2^x} - \alpha^{-2^x})$, and there can

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be no other because when u_0, u_1 are given u_x is absolutely determined.

But furthermore we may invert the above equation by interchanging u_x and u_{x+2}, which gives the equation

$$(u_x + 2u_{x+1})\, u^2_{x+2} - u^2_{x+1} = 0,$$

of which the solution will obviously be $u_x = C\left(P - \frac{1}{P}\right)$, where $P = \alpha^{(\frac{1}{2})^x}$.

Suppose u_0, u_1 to be given; then

$$C\left(\alpha - \frac{1}{\alpha}\right) = u_0, \quad C\left(\alpha^{\frac{1}{2}} - \frac{1}{\alpha^{\frac{1}{2}}}\right) = u_1,$$

and calling $\frac{u_0}{u_1} = 2r$, $\alpha^{\frac{1}{2}} + \frac{1}{\alpha^{\frac{1}{2}}} = 2r$, $\alpha^{\frac{1}{2}} - \frac{1}{\alpha^{\frac{1}{2}}} = 2\sqrt{(r^2-1)}$,

$$C = \frac{u_0}{4r\sqrt{(r^2-1)}}.$$

Hence

$$u_x = \frac{u_0}{4r\sqrt{(r^2-1)}}\left[\{r+\sqrt{(r^2-1)}\}^{(\frac{1}{2})^{x-1}} - \{r-\sqrt{(r^2-1)}\}^{(\frac{1}{2})^{x-1}}\right],$$

has exactly 2^{x-1} values, for the change of $\sqrt{(r^2-1)}$ into $-\sqrt{(r^2-1)}$ changes simultaneously the signs of the numerator and denominator of this fraction. But by the general principle u_x ought to have 2^{x-1} values in terms of u_0, u_1. Hence the above integral is *exhaustive*.

Suppose now we were to write

$$u_x = C(\alpha^{2^x} + \beta^{2^x}) \text{ with } \alpha\beta = 1;$$

for brevity sake call $u_x = f$, $u_{x+1} = g$, $u_{x+2} = h$,

then

$$C(P+Q) = f,$$
$$C(P^2+Q^2) = g,$$
$$C(P^4+Q^4) = h,$$
$$PQ = 1.$$

Hence

$$f^2 = Cg + 2C^2,$$
$$g^2 = Ch + 2C^2,$$
$$C = \frac{f^2-g^2}{g-h},$$
$$f^2 = \frac{f^2-g^2}{g-h} \cdot \frac{2f^2-g^2-gh}{g-h},$$

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or $f^2g^2 - 2f^3gh + f^2h^2 = 2f^4 - 3f^2g^2 + g^4 - f^3gh + g^3h,$

or $f^3h^2 - (g^3 + f^3g)\,h - g^4 + 4f^3g^2 - 2f^4 = 0,$

or $u_x^3u_{x+2}^2 - (u_{x+1}^3 + u_x^3u_{x+1})\,u_{x+2} - u_{x+1}^4 + 4u_x^2u_{x+1}^2 - 2u_x^4 = 0,$

of which the correlative equation is

$$-2u_{x+2}^4 + (4u_{x+1}^3 - u_{x+1}u_x + u_x^2)\,u_{x+2}^2 - u_{x+1}^3u_x - u_{x+1}^4 = 0.$$

A complete solution of the former of these will therefore be

$$u_x = C(\alpha^{2^x} + \beta^{2^x}),$$

and of the latter

$$u_x = C(\alpha^{(\frac{1}{2})^x} + \beta^{(\frac{1}{2})^x}),$$

but neither of these will be an *exhaustive* solution, for in the one the most general value of u_x ought to be a 2^{x-1}-valued function and in the latter a 4^{x-1}-valued function, whereas the actual value is only one-valued in the one case and 2^{x-1}-valued in the other.

Suppose again we write

$$u_x = C(\alpha^{3^x} - \beta^{3^x}), \text{ where } \alpha\beta = 1, \text{ as before,}$$

say $$u_x = C(P - Q), \text{ where } PQ = 1.$$

Then with the same notation as before

$$C(P - Q) = f,$$

$$C(P^3 - Q^3) = g,$$

$$C(P^9 - Q^9) = h,$$

$$\frac{g}{f} - 1 = P^3 + Q^3, \quad \frac{h}{g} - 1 = P^6 + Q^6,$$

$$\frac{h}{g} - 1 = \left(\frac{g}{f} - 1\right)^3 - 3\left(\frac{g}{f} - 1\right),$$

or

$$\frac{h}{g} = \left(\frac{g}{f}\right)^3 - 3\left(\frac{g}{f}\right)^2 + 3,$$

$$f^3h - 3f^3g = g^4 - 3g^3f,$$

$$\frac{h - 3g}{g - 3f} = \frac{g^3}{f^3}.$$

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Whence it follows that the integrals of

$$\frac{u_{x+2}-3u_{x+1}}{u_{x+1}-3u_x}-\frac{u^3_{x+1}}{u^3_x}=0,$$

and of

$$\frac{u^3_{x+2}}{u^3_{x+1}}-\frac{3u_{x+2}-u_{x+1}}{3u_{x+1}-u_x}=0,$$

are respectively

$$u_x=C(\alpha^{3x}-\alpha^{-3x}),$$

and

$$u_x=C(\alpha^{(\frac{1}{3})x}-\alpha^{-(\frac{1}{3})x}),$$

with the understanding that $\alpha^{-\frac{1}{3}}.\alpha^{\frac{1}{3}}=1$.

These integrals are evidently *exhaustive*.

By writing $\sqrt{(-1)}\alpha$, $-\sqrt{(-1)}\alpha^{-1}$ for α, α^{-1} respectively, f, g, h become increased in the ratio of $\sqrt{(-1)}, -\sqrt{(-1)}, \sqrt{(-1)}$, respectively.

Hence the equations

$$\frac{u_{x+2}+3u_{x+1}}{u_{x+1}+3u_x}-\frac{u^3_{x+1}}{u^3_x}=0,$$

and

$$\frac{u^3_{x+2}}{u^3_{x+1}}-\frac{3u_{x+2}+u_{x+1}}{3u_{x+1}+u_x}=0,$$

have for their solutions

$$u_x=C(\alpha^{3x}+\alpha^{-3x}) \text{ and } u_x=C(\alpha^{(\frac{1}{3})x}+\alpha^{-(\frac{1}{3})x}).$$

Hitherto we have been dealing with *homogeneous* uni-solutional equations. It is easy, however, to form non-homogeneous ones by an obvious process. For, if we write

$$u_x=a_1{}^{m^x}+a_2{}^{m^x}+\ldots+a_i{}^{m^x} \ (m \text{ being an integer}),$$

by eliminating between

$$f_0=\Sigma a,\ f_1=\Sigma a^m,\ f_2=\Sigma a^{m^2},\ \ldots f_i=\Sigma a^{m^i},$$

we shall obtain a relation between the f's of the first degree in f_i and of the degree m^i in f_0, corresponding to which there will be a difference equation of the i^{th} order in which the upper extreme degree is unity and the lower one m^i, of which the integral will be the value of u_x above written, and by interchanging u_x, u_{x+1}, u_{x+i} respectively with $u_{x+i}, u_{x+i-1}, \ldots u_x$, another in which the lower degree is unity and the upper one m^i, of which the integral will be

$$u_x=a_1\left(\frac{1}{m}\right)^x+a_2\left(\frac{1}{m}\right)^x+\ldots+a_i\left(\frac{1}{m}\right)^x,$$

each of which equations will evidently be uni-solutional.

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Or, again, if instead of the a's being independent we make their product equal to unity we shall obtain uni-solutional equations of the $(i-1)^{\text{th}}$ instead of the i^{th} order.

Thus *ex. gr.* let

$$u_x = a^{2x} + b^{2x} + c^{2x} \text{ with the condition } abc = 1.$$

Then writing $u_x = f$, $u_{x+1} = g$, $u_{x+2} = h$,

$$f = A + B + C,\ g = A^2 + B^2 + C^2,\ h = A^4 + B^4 + C^4,$$

$$f^2 - g = 2(AB + AC + BC),$$

$$2(g^2 - h) = 4(A^2B^2 + A^2C^2 + B^2C^2),$$

$$= (f^2 - g)^2 - 8f.$$

Hence we obtain the uni-solutional equations

$$2u_{x+2} - u^2_{x+1} - 2u_{x+1}u_x^2 + u_x^4 - 8u_x = 0,$$

$$u^4_{x+2} - 2u_{x+1}u^2_{x+2} - 8u_{x+2} - u^2_{x+1} + 2u_x = 0,$$

of which the integrals are known and are exhaustive.

We may in a similar manner obtain uni-solutional *simultaneous* difference equations.

Thus let $u_x = C(\alpha^{3x} - \beta^{3x})$, $v_x = C'(\alpha^{3x} + \beta^{3x})$,

and call u_x, u_{x+1}, u_{x+2} as before f, g, h,

and v_x, v_{x+1}, v_{x+2} " l, m, n.

Then $\dfrac{g}{f} = P^2 + PQ + Q^2$, $\dfrac{m}{l} = P^2 - PQ + Q^2$,

$$\frac{h}{g} = P^6 + P^3Q^3 + Q^6 \quad \frac{n}{m} = P^6 - P^3Q^3 + Q^6.$$

Hence $\dfrac{h}{g} - \dfrac{n}{m} = \frac{1}{4}\left(\dfrac{g}{f} - \dfrac{m}{l}\right)^3$,

$$\frac{h}{g} + \frac{n}{m} = 2(P^6 + Q^6)$$

$$= 2(P^2 + Q^2)(P^4 - P^2Q^2 + Q^4)$$

$$= 2(P^2 + Q^2)\{(P^2 + Q^2)^2 - 3P^2Q^2\}$$

$$= \frac{1}{2}\left(\frac{g}{f} + \frac{m}{l}\right)\left\{\left(\frac{g}{f} + \frac{m}{l}\right)^2 - 3\left(\frac{g}{f} - \frac{m}{l}\right)^2\right\}$$

$$= -\frac{1}{4}\left(\frac{g}{f} + \frac{m}{l}\right)\left(2\frac{g^2}{f^2} - 8\frac{g}{f}\cdot\frac{m}{l} + 2\frac{m^2}{l^2}\right).$$

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Hence $\frac{h}{g} = \frac{1}{8}\left(-\frac{g^3}{f^3} + 3\frac{g^2}{f^2}\cdot\frac{m}{l} + 9\frac{g}{f}\cdot\frac{m^2}{l^2} - 3\frac{m^3}{l^3}\right)$,

$$\frac{n}{m} = \frac{1}{8}\left(-3\frac{g^3}{f^3} + 9\frac{g^2}{f^2}\cdot\frac{m}{l} + 3\frac{g}{f}\cdot\frac{m^2}{l^2} - \frac{m^3}{l^3}\right).$$

Obviously, when u_0, u_1; v_0, v_1 are given, each u_x and v_x deduced from the above system of equations has only one value, so that their exhaustive integral will be

$$u_x = C(\alpha^{3x} - \beta^{3x}),\ v_x = C'(\alpha^{3x} + \beta^{3x}).$$

The related system found by interchanging f with h and l with n will be

$$\frac{f}{g} = \frac{1}{8}\left(-\frac{g^3}{h^3} + 3\frac{g^2}{h^2}\cdot\frac{m}{n} + 9\frac{g}{h}\cdot\frac{m^2}{n^2} - 3\frac{m^3}{n^3}\right),$$

$$\frac{l}{m} = \frac{1}{8}\left(-3\frac{g^3}{h^3} + 9\frac{g^2}{h^2}\cdot\frac{m}{n} + 3\frac{g}{h}\cdot\frac{m^2}{n^2} - \frac{m^3}{n^3}\right).$$

When f, g; l, m are given the system $\frac{1}{h}, \frac{1}{n}$ may be found by solving an equation of the 9th degree. Hence, when u_0, u_1; v_0, v_1 are given, u_2, v_2 will have 9; u_3, v_3, 81, and in general u_x, v_x will have $3^{2(x-1)}$ values which will correspond to the $3^{x-1}.3^{x-1}$ values of u_x, v_x.

The apparent number of values of each of these is $(3^x)^2$, which, however, must be reducible to $3^{x-1}.3^{x-1}$ when expressed in terms of the two initial values of u and of v, similarly to what was noticed at the outset on the reduction of the apparent multiplicity 2^x to a multiplicity 2^{x-1}.

In fact, we we write

$$u_x = C(\alpha^{(\frac{1}{3})x} - \beta^{(\frac{1}{3})x}),\ v_x = C'(\alpha^{(\frac{1}{3})x} + \beta^{(\frac{1}{3})x}),$$

$$u_0 = C(\alpha - \beta),\ u_1 = C(\alpha^{\frac{1}{3}} - \beta^{\frac{1}{3}});\ v_0 = C'(\alpha + \beta),\ v_1 = C'(\alpha^{\frac{1}{3}} + \beta^{\frac{1}{3}}),$$

$$\alpha^{\frac{2}{3}} + \alpha^{\frac{1}{3}}\beta^{\frac{1}{3}} + \beta^{\frac{2}{3}} = \frac{u_0}{u_1},\quad \alpha^{\frac{2}{3}} - \alpha^{\frac{1}{3}}\beta^{\frac{1}{3}} + \beta^{\frac{2}{3}} = \frac{v_0}{v_1},$$

$$\alpha^{\frac{1}{3}}\beta^{\frac{1}{3}} = \tfrac{1}{2}\left(\frac{u_0}{u_1} - \frac{v_0}{v_1}\right),$$

$$\alpha^{\frac{2}{3}} + \beta^{\frac{2}{3}} = \tfrac{1}{2}\left(\frac{u_0}{u_1} + \frac{v_0}{v_1}\right),$$

$$\alpha^{\frac{1}{3}} - \beta^{\frac{1}{3}} = \sqrt{\left\{\tfrac{1}{2}\left(3\frac{v_0}{v_1} - \frac{u_0}{u_1}\right)\right\}},$$

$$\alpha^{\frac{1}{3}} + \beta^{\frac{1}{3}} = \sqrt{\left\{\tfrac{1}{2}\left(3\frac{u_0}{u_1} - \frac{v_0}{v_1}\right)\right\}},$$

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$$C=\frac{u_1}{\sqrt{\left\{\frac{1}{2}\left(\frac{3v_0}{v_1}-\frac{u_0}{u_1}\right)\right\}}},\quad C'=\frac{v_1}{\sqrt{\left\{\frac{1}{2}\left(\frac{3u_0}{u_1}-\frac{v_0}{v_1}\right)\right\}}},$$

$$\alpha^{\frac{1}{2}}=\sqrt{\left\{\frac{1}{8}\left(\frac{3u_0}{u_1}-\frac{v_0}{v_1}\right)\right\}}+\sqrt{\left\{\frac{1}{8}\left(\frac{3v_0}{v_1}-\frac{u_0}{u_1}\right)\right\}},$$

$$\beta^{\frac{1}{2}}=\sqrt{\left\{\frac{1}{8}\left(\frac{3u_0}{u_1}-\frac{v_0}{v_1}\right)\right\}}-\sqrt{\left\{\frac{1}{8}\left(\frac{3v_0}{v_1}-\frac{u_0}{u_1}\right)\right\}},$$

and thus for the final values of u_x and v_x, we find

$$u_x=\frac{u_1}{\sqrt{\left\{\frac{1}{2}\left(\frac{3v_0}{v_1}-\frac{u_0}{u_1}\right)\right\}}}\times\left\{\left[\sqrt{\left\{\frac{1}{8}\left(\frac{3u_0}{u_1}-\frac{v_0}{v_1}\right)\right\}}+\sqrt{\left\{\frac{1}{8}\left(\frac{3v_0}{v_1}-\frac{u_0}{u_1}\right)\right\}}\right]^{(\frac{1}{2})^{x-1}}-\left[\sqrt{\left\{\frac{1}{8}\left(\frac{3u_0}{u_1}-\frac{v_0}{v_1}\right)\right\}}-\sqrt{\left\{\frac{1}{8}\left(\frac{3v_0}{v_1}-\frac{u_0}{u_1}\right)\right\}}\right]^{(\frac{1}{2})^{x-1}}\right\},$$

$$v_x=\frac{v_1}{\sqrt{\left\{\frac{1}{2}\left(\frac{3u_0}{u_1}-\frac{v_0}{v_1}\right)\right\}}}\times\left\{\left[\sqrt{\left\{\frac{1}{8}\left(\frac{3u_0}{u_1}-\frac{v_0}{v_1}\right)\right\}}+\sqrt{\left\{\frac{1}{8}\left(\frac{3v_0}{v_1}-\frac{u_0}{u_1}\right)\right\}}\right]^{(\frac{1}{2})^{x-1}}+\left[\sqrt{\left\{\frac{1}{8}\left(\frac{3u_0}{u_1}-\frac{v_0}{v_1}\right)\right\}}-\sqrt{\left\{\frac{1}{8}\left(\frac{3v_0}{v_1}-\frac{u_0}{u_1}\right)\right\}}\right]^{(\frac{1}{2})^{x-1}}\right\},$$

each of which is unaffected by a change in the signs of the square roots, so that u_x and v_x are seen to be $3^{2^{x-1}}$ valued functions, and (u_x, v_x) a 9^{x-1} valued system, as should be the case for an exhaustive solution of the last written difference equations.

Let us tentatively go a step further in the same direction and suppose that we are given

$$u_x=C(a^{5^x}-b^{5^x}),\quad v_x=C'(a^{5^x}+b^{5^x}),$$

and use $f, g, h; l, m, n$ in the same way as before, and furthermore, write

$$\frac{1}{2}\left(\frac{g}{f}+\frac{m}{l}\right)=L,\quad \frac{1}{2}\left(\frac{h}{g}+\frac{n}{m}\right)=N,$$

$$\frac{1}{2}\left(\frac{g}{f}-\frac{m}{l}\right)=M,\quad \frac{1}{2}\left(\frac{h}{g}-\frac{n}{m}\right)=P,$$

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we shall find

$$L = A^4 + A^2B^2 + B^4, \quad N = A^{20} + A^{10}B^{10} + B^{20},$$
$$M = AB(A^2 + B^2), \quad P = A^5B^5(A^{10} + B^{10}),$$

(where $A = a^{5^2}$ and $B = b^{5^2}$).

Let $A^2 + B^2 = \lambda, \ AB = \mu.$

Then $L = \lambda^2 - \mu^2, \ M = \lambda\mu,$

and it will be seen that

$$N = (\lambda^5 - 5\lambda^3\mu^2 + 5\lambda\mu^4)^2 - \mu^{10},$$
$$P = \lambda^5\mu^5 - 5\lambda\mu^7(\lambda^2 - \mu^2).$$

For $A^6 + B^6 = \lambda^3 - 3\lambda\mu^2,$

and consequently

$$\lambda^5 = A^{10} + B^{10} + 5A^2B^2(A^6 + B^6) + 10A^4B^4(A^2 + B^2)$$
$$= A^{10} + B^{10} + 5\mu^2(\lambda^3 - 3\lambda\mu^2) + 10\lambda\mu^4,$$

i.e. $A^{10} + B^{10} = \lambda^5 - 5\lambda^3\mu^2 + 5\lambda\mu^4.$

The above values of N and P (remembering that $AB = \mu$) are found by substituting the expression just obtained for $A^{10} + B^{10}$ in

$$N = A^{20} + A^{10}B^{10} + B^{20},$$
$$P = A^5B^5(A^{10} + B^{10}).$$

From $P = \lambda^5\mu^5 - 5\lambda\mu^7(\lambda^2 - \mu^2),$

(remembering that $\lambda\mu = M, \ \lambda^2 - \mu^2 = L$), we obtain

$$P = M^5 - 5LM\mu^6.$$

Hence
$$\left.\begin{aligned} \frac{M^5 - P}{5LM^4} &= \frac{\mu^3}{\lambda^3} \\ \frac{L}{M} &= \frac{\lambda}{\mu} - \frac{\mu}{\lambda} \end{aligned}\right\}.$$

From these equations we obtain by elimination

$$[1]\ldots\left(\frac{P}{M}\right)^2 + (2M^4 + 15L^2M^2 + 5L^4)\frac{P}{M} + M^4(M^4 - 10L^2M^2 + 5L^4) = 0.$$

Similarly by an elimination into the details of which it is unnecessary to enter we obtain

$$[2]\ldots\ldots 3LMP + (L^2 + M^2)N = L(L^2 - 2M^2)(L^2 - M^2)^2,$$

which gives a *linear* relation between N and P.

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Equations (1) and (2) form a non-uni-solutional system of which (as also of its inverse) we are in possession of one complete integral, and I have some grounds for suspecting that it may be possible to obtain from this a second (so-called *indirect*) integral, but am unable for the present to pursue the subject further.

The preceding investigation originated in my attention happening to be called to Vieta's well known theorem for approximating to the *Archimedean* constant (π) by means of an indefinite product of cosines of continually bisected angles. The implied connexion of ideas will become apparent when one considers that any one of such cosines may be expressed as a sum of two binary exponentials with $\frac{1}{2}$ for the first index, and that thus Vieta's theorem (although presumably obtained by him as a very simple consequence of the method of exhaustions) in its essence depends on the integrability of a uni-solutional difference equation of the 2nd order of the form treated of at the outset of this paper.

New College, Oxford,
Dec. 11*th*, 1888.

DEPRESSION OF DIFFERENTIAL EQUATIONS.

By *Lt.-Col. Allan Cunningham, R.E.*, Fellow of King's Coll., Lond., &c.

27. THIS paper is only a supplement to the paper on Depression of Differential Equations in pp. 118 to 145 of Vol. XVII. It contains the detail of three *additional* modes of solution (by depression) of the differential equation of a conic,

$$9y''^2y' - 45y'y''y''' + 40y'''^3 = 0,$$

not quite so simple as those given at pp. 140 to 144 of above. The notation and procedure are as in that paper, *q. v.*

As this equation is clear of x, y, y', it may be depressed one order by depression i., and will still be clear of the new variables x_1, y_1, and will still be multiply homogeneous. It may therefore be further depressed (Art. 16) two orders by a double application of depression ii. without destroying the homogeneity; also the depression for homogeneity of order ∞ may be effected before or after the second of depressions ii. (Art. 9).

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Hence (see also Art. 12), the five depressions required in all may be effected in any of the following orders:—

COURSE 4°. *Depressions* i., ii., ii., iii. (ν arbitrary), iii. ($\nu=1$).

„ 5°. „ i., ii., ii., iv., iii. ($\nu=-1$).

„ 6°. „ i., ii., iv., ii., iii. ($\nu=2$).

The depressions are shown step by step in Table X. below, and the integrations also step by step in Table XI. To facilitate the integration in Step II., Course 4°, which seems impossible with the general value of ν, the special values $\nu=\frac{1}{3}$ or $\frac{2}{3}$ have been adopted in the integration.

DEPRESSION OF DIFFERENTIAL EQUATION OF CONIC.

TABLE X.

$$9y'^2y'-45y''y'''y''+40y'''^3=0,$$

(an equation wanting x, y, y', and multiply homogeneous).

STEP I. *Depression* i. (take $y'=y_1$, $x=x_1$).

$$9y_1^2y_1''-45y_1'y_1''y_1'''+40y_1'''^3=0,$$

(an equation wanting x_1, y_1, and multiply homogeneous).

STEP II. *Depression* ii. (take $y_1=x_2$, $y_1'=y_2$).

$$9y_2^3y_2'''-9y_2y_2'y_2''+4y_2'^3=0,$$

(an equation wanting x_2, and multiply homogeneous).

There are now three Courses (Nos. 4°, 5°, 6°) open as explained above.

COURSE 4°. STEP III. *Depression* ii. (take $y_2=x_3$, $y_2'=y_3$).

$$9x_3^2(y_3y_3''+y_3'^2)-9x_3y_3y_3'+4y_3^2=0,$$

(an equation possessing multiple homogeneity).

STEP IV. *Depression* iii. (ν arbitrary).

$$x_4y_4y_4'+y_4^2+(4\nu-2)x_4y_4+(2\nu^2-2\nu+\tfrac{4}{9})x_4^2=0,$$

(an equation homogeneous in order 1).

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STEP V. *Depression* iii. ($\nu = 1$).

$$x_5 y_5 + 2\,(x_5 + \nu - \tfrac{2}{3})\,(x_5 + \nu - \tfrac{1}{3}) = 0,$$

(the final depressed equation wherein ν is still arbitrary).

COURSE 5°. STEP III. *Depression* ii. ($y_3 = x_3$, $y_3' = y_3$).

$$9x_3^2\,(y_3 y_3'' + y_3'^2) - 9x_3 y_3 y_3' + 4y_3^2 = 0,$$

(an equation possessing multiple homogeneity).

STEP IV. *Depression* iv.

$$x_4^2\,(y_4' + 2y_4^2) - x_4 y_4 + \tfrac{4}{9} = 0.$$

(an equation homogeneous in order $\nu = -1$).

STEP V. *Depression* iii. ($\nu = -1$).

$$y_5 + 2\,(x_5 - \tfrac{2}{3})\,(x_5 - \tfrac{1}{3}) = 0,$$

(the final depressed equation).

COURSE 6°. STEP III. *Depression* iv.

$$9y_3'' + 18y_3 y_3' + 4y_3^3 = 0,$$

(an equation wanting x_3, and homogeneous in order -1).

STEP IV. *Depression* ii.

$$y_4 y_4' + 2x_4 y_4 + \tfrac{4}{9}x_4^3 = 0,$$

(an equation homogeneous in order $\nu = 2$).

STEP V. *Depression* iii. ($\nu = 2$).

$$x_5 y_5 + 2\,(x_5 + \tfrac{2}{3})\,(x_5 + \tfrac{1}{3}) = 0,$$

(the final depressed equation).

DIFFERENTIAL EQUATION OF CONIC.

TABLE XI.

Integration of final depressed equation of Course 4°, 5°, or 6°.

COURSE 4°. *Depressions* i., ii., ii., iii. ($\nu = \nu$), iii. ($\nu = 1$).

Final depressed equation.

$$x_5 y_5 + 2\,(x_5 + \nu - \tfrac{2}{3})\,(x_5 + \nu - \tfrac{1}{3}) = 0.$$

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STEP I. *Depression* iii. ($\nu = 1$) *reversed.*

Formulæ. $x_5 = \frac{y_4}{x_4}$, $y_5 = x_4 \frac{dx_5}{dx_4}$.

Equation in x_4, x_5. $cx_4^{\frac{3}{5}} = \frac{(x_5 + \nu - \frac{2}{5})^{\nu - \frac{2}{5}}}{(x_5 + \nu - \frac{1}{5})^{\nu - \frac{1}{5}}}$.

Equation in x_4, y_4. $cx_4^{\frac{1}{5}} = \frac{\{y_4 + (\nu - \frac{2}{5})\, x_4\}^{\nu - \frac{2}{5}}}{\{y_4 + (\nu - \frac{1}{5})\, x_4\}^{\nu - \frac{1}{5}}}$.

N.B. In these latter two results ν is arbitrary.

STEP II. *Depression* iii. (ν *arbitrary*) *reversed.*

Equation in x_4, y_4 *solved for* y_4.

Make $\nu = \frac{1}{5}$ or $\frac{2}{5}$,

therefore $y_4 = \frac{1}{cx_4} \pm \frac{1}{5} x_4$.

Formulæ. $x_4 = x_3^{-\nu} y_3$, $y_4 = x_3 \frac{dx_3}{dx_3}$.

Equation in x_3, x_4. $cx_3^{\pm\frac{2}{5}} = C \pm x_4^{\frac{2}{5}}$.

Equation in x_3, y_3. $y_3 = x_3^{\frac{1}{2}} \sqrt{(A_3 x_3^{\frac{1}{5}} + B_3 x_3^{-\frac{1}{5}})}$.

STEP III. *Depression* ii. *reversed.*

Formulæ. $x_2 = y_2$, $y_2 = \frac{dy_2}{dx_2}$.

Equation in x_2, y_2. $y_2 = (A_2 + B_2 x_2 + C_2 x_2^2)^{\frac{1}{2}}$.

STEP IV. *Depression* ii. *reversed.*

Formulæ. $x_2 = y_1$, $y_2 = \frac{dy_1}{dx_1}$.

Equation in x_1, y_1. $y_1 - A = \frac{k(C_1 x + D_1)}{\sqrt{\{1 - (C_1 x + D_1)^2\}}}$.

STEP V. *Depression* i. *reversed.*

Formulæ. $x_1 = x$, $y_1 = y'$.

Final equation in x, y. $y = Ax + B \pm \sqrt{(Cx^2 + Dx + E)}$,

which is the required complete primitive, and is the algebraic equation of a conic.

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A CORRESPONDENCE OF CONFOCAL CARTESIANS WITH THE RIGHT LINES OF A HYPERBOLOID.

By Prof. *Cayley*.

TAKE α, β, γ arbitrary, A, B, $C = \beta - \gamma$, $\gamma - \alpha$, $\alpha - \beta$ (so that $A + B + C = 0$), and writing ρ, σ, τ for rectangular coordinates, consider the hyperboloid

$$A\rho^2 + B\sigma^2 + C\tau^2 + ABC = 0.$$

Let ρ_0, σ_0, τ_0 be the coordinates of a point on the surface $(A\rho_0^2 + B\sigma_0^2 + C\tau_0^2 + ABC = 0)$, the equations of a line through this point are ρ, σ, $\tau = \rho_0 + f\Omega$, $\sigma_0 + g\Omega$, $\tau_0 + h\Omega$ (Ω indeterminate); and if this lie on the surface, we have

$$A\rho_0 f + B\sigma_0 g + C\tau_0 h = 0,$$
$$Af^2 + Bg^2 + Ch^2 = 0;$$

which equations determine the ratios $f : g : h$; the equations give

$$(A\rho_0 f + B\sigma_0 g)^2 = C\tau_0^2 . Ch^2, \quad = -C\tau_0^2 (Af^2 + Bg^2);$$

that is

$$(A^2\rho_0^2 + AC\tau_0^2)f^2 + 2AB\rho_0\sigma_0 fg + (B^2\sigma_0^2 + BC\tau_0^2)g^2 = 0,$$

whence

$$\{(B^2\sigma_0^2 + BC\tau_0^2)g + AB\rho_0\sigma_0 f\}^2$$
$$= \{A^2B^2\rho_0^2\sigma_0^2 - (A^2\rho_0^2 + AC\tau_0^2)(B^2\sigma_0^2 + BC\tau_0^2)\}f^2,$$
$$= -ABC(A\rho_0^2 + B\sigma_0^2 + C\tau_0^2)\tau_0^2 f^2,$$
$$= A^2B^2C^2\tau_0^2 f^2;$$

that is

$$\{(B\sigma_0^2 + C\tau_0^2)g + A\rho_0\sigma_0 f\}^2 = A^2C^2\tau_0^2 f^2,$$

or say

$$(B\sigma_0^2 + C\tau_0^2)g + A(\rho_0\sigma_0 \pm C\tau_0)f = 0,$$

which equation, together with $A\rho_0 f + B\sigma_0 g + C\tau_0 h = 0$, determines the ratios $f : g : h$. We have thus the two lines through the point $(\rho_0, \sigma_0, \tau_0)$.

But the equations of the line may be conveniently represented in a different form; writing the equation first obtained in the form

$$\sigma_0(B\sigma_0 g + A\rho_0 f) + C\tau_0^2 g \pm AC\tau_0 f = 0,$$

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this is $$-\sigma_0 C\tau_0 h + C\tau_0^2 g \pm AC\tau_0 f = 0,$$

viz. $$-h\sigma_0 + g\tau_0 \pm Af = 0;$$

and we have the like equations

$$-f\tau_0 + h\rho_0 \pm Bg = 0,$$
$$-g\rho_0 + f\sigma_0 \pm Ch = 0,$$

where the sign is the same in each of the three equations.

The equations of the line on the surface may be written

$$\begin{aligned} . \quad & h\sigma \quad && -g\tau - h\sigma_0 + g\tau_0 = 0, \\ -h\rho \quad & . \quad && +f\tau - f\tau_0 + h\rho_0 = 0, \\ g\rho \quad & -f\sigma \quad && . \quad -g\rho_0 + f\sigma_0 = 0, \\ (h\sigma_0 - g\tau_0)\rho & + (f\tau_0 - h\rho_0)\sigma + (g\sigma_0 - f\rho_0)\tau && = 0; \end{aligned}$$

and hence from the foregoing three equations, taking the sign −, we have

$$\begin{aligned} . \quad & h\sigma \quad - g\tau \quad + Af = 0, \\ -h\rho \quad & . \quad + f\tau \quad + Bg = 0, \\ g\rho \quad & - f\sigma \quad . \quad + Ch = 0, \\ -Af\rho \quad & - Bg\sigma - Ch\tau \quad . \quad = 0, \end{aligned}$$

where $Af^2 + Bg^2 + Ch^2 = 0$, for the equations of a line on the surface.

In like manner taking the sign +, and for f, g, h writing new values f', g', h', we have

$$\begin{aligned} . \quad & h'\sigma \quad - g'\tau \quad - Af' = 0, \\ -h'\rho \quad & . \quad + f'\tau \quad - Bg' = 0, \\ g'\rho \quad & - f'\sigma \quad . \quad - Ch' = 0, \\ Af'\rho \quad & + Bg'\sigma + Ch'\tau \quad . \quad = 0, \end{aligned}$$

where $Af'^2 + Bg'^2 + Ch'^2 = 0$, for the equations of a line on the surface.

The two systems of equations evidently belong to the lines of the two different kinds respectively. Writing for shortness P, Q, $R = gh' + g'h$, $hf' + h'f$, $fg' + f'g$, the two lines in fact intersect in a point, the coordinates say $(\rho_0, \sigma_0, \tau_0)$ whereof are $= \Theta QR$, ΘRP, ΘPQ, where

$$\Theta = \frac{A}{g^2h'^2 - g'^2h^2} = \frac{B}{h^2f'^2 - h'^2f^2} = \frac{C}{f^2g'^2 - f'^2g^2},$$

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the three expressions for Θ being equal to each other in virtue of the equations $Af^2 + Bg^2 + Ch^2 = 0$, $Af'^2 + Bg'^2 + Ch'^2 = 0$.

Take now, in a plane, P, Q, R points on any line, say the axis of x, at distances α, β, γ from the origin, then for a point of the plane, coordinates (x, y), if ρ, σ, τ be the distances of the point from these three points, or say foci, we have

$$\rho^2 = (x - \alpha)^2 + y^2,$$

$$\sigma^2 = (x - \beta)^2 + y^2,$$

$$\tau^2 = (x - \gamma)^2 + y^2;$$

and if as before A, B, $C = \beta - \gamma, \gamma - \alpha, \alpha - \beta$, we thence have

$$A\rho^2 + B\sigma^2 + C\tau^2 + ABC = 0.$$

A point, coordinates (ρ, σ, τ), of the hyperboloid thus corresponds to a point in the plane, distances ρ, σ, τ from the three foci R, S, T respectively; and to any line

$$\begin{aligned} . \quad & h\sigma - g\tau + Af = 0, \\ -h\rho \quad & . + f\tau + Bg = 0, \\ g\rho - f\sigma \quad & . + Ch = 0, \\ -Af\rho - Bg\sigma - Ch\tau \quad & . = 0, \end{aligned}$$

corresponds the Cartesian represented by these linear equations. Similarly to the line represented by the other system of equations

$$\begin{aligned} . \quad & h'\sigma - g'\tau - Af' = 0, \\ -h'\rho \quad & . + f'\tau - Bg' = 0, \\ g'\rho - f'\sigma \quad & . - Ch' = 0, \\ Af'\rho + Bg'\sigma + Ch'\tau \quad & . = 0, \end{aligned}$$

corresponds the Cartesian represented by these equations; the two curves intersect in the point $\rho_0, \sigma_0, \tau_0 = \Theta QR, \Theta RP, \Theta PQ$, corresponding to the intersection of the lines on the hyperboloid; and moreover, *quà* confocal Cartesians, they intersect at right angles.

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ESTABLISHMENT OF THE FUNDAMENTAL PROPERTIES OF QUATERNIONS.

By *Alex. McAulay*, Ormond College, Melbourne.

It is desirable to smooth the path of the beginner in Quaternions as much as possible, if only to induce more to study the subject. There is little doubt that the establishment of the fundamental properties of Quaternions in Tait's treatise on the subject (Chap. II.) proves difficult to most beginners, and effectually dissuades many waverers from continuing the subject. In Chap. III. of Kelland and Tait's treatise we are introduced to the general subject in a piecemeal way that does not give complete satisfaction to the mathematical mind.

Can simplicity of proof be united with a sufficiently comprehensive definition of a Quaternion? The object of the present paper is to indicate two methods which will be found, I think, simpler than Tait's and more suitable to the subject than Kelland's.

We start with the same meaning attached to the term Quaternion as Tait, whence the following definition: *A Quaternion is an operator which turns any vector perpendicular to a definite axis through a definite angle round the axis, and changes its length in a definite ratio.* Thus, a quaternion is completely specified by the direction of its axis and two numbers, one to denote the angle mentioned and the other the ratio. At present we have no meaning for $q\alpha$ if α be a vector *not* perpendicular to the axis of q, or, as we may put it, not in the plane of q.

Suppose now the vector α is in the plane of the quaternion r and $r\alpha$ is in the plane of the quaternion q, then the product qr is defined as the quaternion which changes α into $q(r\alpha)$. Again, if α is in the planes both of q and r, the sum $q+r$ is defined as the quaternion which changes α into $q\alpha+r\alpha$.

The conjugate Kq of any quaternion q is defined as a quaternion whose axis is in the opposite direction to that of q, but is otherwise the same as the latter. It is quite easy to prove, as does Tait, that

$$K(qr)=KrKq \quad \ldots\ldots\ldots\ldots\ldots\ldots(1),$$

$$K(q+r)=Kq+Kr \quad \ldots\ldots\ldots\ldots\ldots\ldots(2).$$

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The fundamental properties of Quaternions may be thus summarised: p, q, r, s being any four quaternions,

$$q+r=r+q \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3),$$

$$p+(q+r)=(p+q)+r \ldots\ldots\ldots\ldots\ldots\ldots(4),$$

$$p.qr=pq.r \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(5),$$

$$(p+q)(r+s)=pr+ps+qr+qs\ldots\ldots\ldots\ldots(6).$$

Eq. (3) presents no difficulty; eq. (4) is, I believe, nowhere explicitly proved by Tait. Hamilton, in his *Elements of Quaternions*, proves this in a very instructive manner. [I am not able at present to obtain a copy of this work, and therefore cannot give the exact reference]. He first defines the scalar part Sq and the vector or quadrantal part Vq of any quaternion q by the equations

$$2\,Sq=q+Kq \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(7),$$

$$2\,Vq=q-Kq \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(8);$$

whence, since all the quaternions here involved are co-planar,

$$q=Sq+Vq \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(9).$$

[Observe that by these definitions Sq is a pure scalar, *i.e.* an ordinary algebraic magnitude; and Vq is a quaternion whose angle, being a right angle, is fixed so that Vq has three and only three disposable constants, two of which go to specify *a direction* just as in the case of a vector]. He then shows by an easy geometrical interpretation that

$$S(q+r)=Sq+Sr \ldots\ldots\ldots\ldots\ldots\ldots(10),$$

$$V(q+r)=Vq+Vr\ldots\ldots\ldots\ldots\ldots\ldots(11),$$

where Sq and Sr being ordinary algebraic quantities are added in the ordinary sense, and where Vq and Vr are added *by the law of addition of vectors.* [It may be noticed in passing that hereby Vq is proved to be strictly a vector, since the sole limitation imposed on vectors is this law of addition. Thus vectors can always be regarded as particular cases of quaternions]. From eqs. (10) and (11) it at once follows that

$$S\Sigma q=\Sigma Sq \ldots\ldots\ldots\ldots\ldots\ldots\ldots(12),$$

$$V\Sigma q=\Sigma Vq \ldots\ldots\ldots\ldots\ldots\ldots\ldots(13),$$

whence by eq. (9)

$$\Sigma q=\Sigma Vq+\Sigma Sq \ldots\ldots\ldots\ldots\ldots\ldots(14),$$

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which proves eq. (4), since vector addition is distributive. The geometrical proofs involved in the above process can easily be applied by the reader who cannot obtain a copy of the Elements.

Eqs. (5) and (6) remain to be proved, and our present object is to obtain simple proofs of these.

We have seen that a quaternion is completely specified by the direction of its axis, the size of its angle and the ratio of its extension. Just as a force is sometimes represented by a straight line let us take as a model of a quaternion a combined rotation and expansion (which may be supposed to operate on any geometrical system), the axis of rotation being the axis of the quaternion, the angle of rotation being *twice* the angle of the quaternion and the ratio of expansion being equal to the extension or what is called the tensor part of the quaternion. The rotation and the combined rotation-expansion thus corresponding to any quaternion q we may call shortly the rotation and rotation-expansion of q respectively. We may now prove that the rotation-expansion of the quaternion qr, q and r being any two quaternions is that obtained by imposing the rotation-expansion of q on that of r.

For let OB be the line of intersection of the planes of q and r, and suppose r changes vector OA into OB and q changes OB into OC. Then by definition qr changes OA into OC. Let OA, OB, OC cut the unit sphere which has O for centre in A', B', C' respectively. Let the poles on this sphere of the arcs $A'B'$, $B'C'$ and $A'C'$ (not $C'A'$ observe) be R, Q, P respectively, so that OQ, OR, OP are the axes of q, r, and qr respectively. By the properties of polar triangles

$$\text{angle } PRQ = A'B', \quad PQR = B'C', \quad \text{and} \quad RPQ = \pi - A'C'.$$

Let P' be chosen so that P and P' are placed symmetrically with regard to QR. Then angle $PRP' = 2A'B'$, so that the rotation of r brings P to P'. Similarly the rotation of q brings P' back to P. Thus the result of imposing the rotation of q on that of r is to get a rotation round OP, that is, round the axis of qr. In the first rotation R is not moved. In the second rotation suppose it comes to R'. Thus angle $R'QR = 2B'C' = 2PQR$, so that R' and R are symmetrically placed with regard to QP. Therefore the angle $R'PR$ of the resultant rotation $= 2(\pi - RPQ) = 2A'C'$. Thus the result of imposing the rotation of q on that of r is to obtain the rotation of qr. The proposition at once

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follows that the rotation-expansion of qr is that obtained by making successively the rotation-expansions of r and q. It now further follows that the rotation-expansions of $p.qr$ and $pq.r$ are exactly the same, being that rotation-expansion obtained by making successively those of r, q and p. Thus $p.qr = pq.r$, which is equation (5).

Eq. (6) can be proved in a manner analogous to the above, but though the process is really quite simple the explanation is long, so we will prove it by aid of eq. (5). If the planes of q, r and p intersect in a common line it is plain by mere interpretation that

$$(q+r)\,p = qp + rp.$$

If these planes be not collinear let the planes of q and r intersect in OA, those of qp and rp in OB, and those of p and OAB in OC, and let p change the vector OD into OC. Let s be the quaternion which changes OB into OD, so that ps changes OB into OC. Thus the planes of ps, q and r intersect in OA, and those of qp, rp and s in OB. Therefore by the last equation and eq. (5)

$$\begin{aligned}(q+r)\,ps &= q.ps + r.ps \\ &= qp.s + rp.s \\ &= (qp+rp)\,s.\end{aligned}$$

Operating then on s^{-1},

$$(q+r)\,p = qp + rp.$$

Taking the conjugate, we see by eqs. (1) and (2) that

$$p\,(q+r) = pq + pr.$$

Combining these results we get eq. (6).

We proceed now to a totally different method of proving (5) and (6).

Hitherto I have purposely introduced as few forms of notation as possible, but the following method is much shortened on paper, though perhaps not quite so fully grasped by the beginner, by adopting the usual notation β/α for the quaternion which changes α into β.

The definition of qr may now be put in the form

$$\frac{\gamma}{\beta}\cdot\frac{\beta}{\alpha} = \frac{\gamma}{\alpha} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (15).$$

Let q, r be any two quaternions and x a scalar. We proceed (cf. Tait's *Quaternions*, § 81 p. 41) to prove that

$$(x+q)\,r = xr + qr.$$

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Let the planes of q and r intersect in the vector β, and let

$$q = \frac{\gamma}{\beta}, \quad r = \frac{\beta}{\alpha}.$$

Then, by the definition of addition of quaternions,

$$x + q = \frac{x\beta + \gamma}{\beta},$$

therefore
$$(x+q)\,r = \frac{x\beta+\gamma}{\beta}\,\frac{\beta}{\alpha} = \frac{x\beta+\gamma}{\alpha}$$
$$= x\frac{\beta}{\alpha} + \frac{\gamma}{\alpha} \text{ (by def. again)}$$
$$= xr + qr.$$

Taking the conjugate, we have by eqs. (1) and (2)

$$r\,(x+q) = rx + rq \ldots\ldots\ldots\ldots\ldots\ldots (16).$$

We proceed to the proof of eq. (6). Let the planes of q, r; r, p; and p, q intersect in the vectors α, β, γ respectively. Then we may put

$$p = x_1\frac{\alpha}{\gamma} + y_1, \quad q = x_2\frac{\beta}{\gamma} + y_2,$$

x_1, y_1, x_2, y_2 being scalars. Since both α and β are in the plane of r, we have

$$r\,\frac{x_1\alpha + x_2\beta}{\gamma} = \frac{r\,(x_1\alpha + x_2\beta)}{\gamma},$$

and, for the same reason, we have

$$r\,(x_1\alpha + x_2\beta) = r\,(x_1\alpha) + r\,(x_2\beta),$$

so that
$$r\,\frac{x_1\alpha + x_2\beta}{\gamma} = \frac{r\,(x_1\alpha)}{\gamma} + \frac{r\,(x_2\beta)}{\gamma},$$

or
$$r\,\frac{x_1\alpha + x_2\beta}{\gamma} = r\,\frac{x_1\alpha}{\gamma} + r\,\frac{x_2\beta}{\gamma} \ldots\ldots\ldots\ldots (17).$$

But
$$r\,(p+q) = r\left(x_1\frac{\alpha}{\gamma} + y_1 + x_2\frac{\beta}{\gamma} + y_2\right)$$
$$= r\left\{\frac{x_1\alpha + x_2\beta}{\gamma} + (y_1 + y_2)\right\}$$
$$= r\,\frac{x_1\alpha + x_2\beta}{\gamma} + ry_1 + ry_2 \quad \text{by eq. (16)}$$
$$= r\,\frac{x_1\alpha}{\gamma} + ry_1 + r\,\frac{x_2\beta}{\gamma} + ry_2 \quad \text{by eq. (17)}$$
$$= rp + rq \quad \text{by eq. (16).}$$

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Taking the conjugate and combining the result with this, we get eq. (6).

Eq. (5) is now quite easy to establish. For, taking p, q, r, α, β, γ as above, we may put

$$p = x_1 \frac{\beta}{\gamma} + y_1 = x_1 q_1 + y_1 \text{ say},$$

$$q = x_2 \frac{\gamma}{\alpha} + y_2 = x_2 q_2 + y_2,$$

$$r = x_3 \frac{\alpha}{\beta} + y_3 = x_3 q_3 + y_3.$$

Thus $$p \cdot qr = x_1 x_2 x_3 q_1 \cdot q_2 q_3 + Q \text{ say},$$

$$pq \cdot r = x_1 x_2 x_3 q_1 q_2 \cdot q_3 + Q,$$

Q being the same in both these expressions, as can be seen by applying eq. (6). Now

$$q_1 \cdot q_2 q_3 = 1 = q_1 q_2 \cdot q_3,$$

therefore $$p \cdot qr = pq \cdot r.$$

Of the above methods the former will probably commend itself to the pure geometer, and the latter to the mathematical analyst. Hence the necessity for both.

February 29, 1888.

NOTE ON THE SOLUTION OF A HOMOGENEOUS DIFFERENTIAL EQUATION.

By *A. R. Forsyth.*

THE following method of integrating the equation

$$\frac{dy}{dx} = \frac{b_0 y^n + b_1 y^{n-1} x + b_2 y^{n-2} x^2 + \ldots + b_n x^n}{a_0 y^n + a_1 y^{n-1} x + a_2 y^{n-2} x^2 + \ldots + a_n x^n} = \frac{V(y, x)}{U(y, x)},$$

is given because in it use is made of Bezout's process of elimination.

Evidently V and U may be supposed to have no common factor.

Writing $$\frac{dy}{V(y, x)} = \frac{dx}{U(y, x)} = dt,$$

we have $$\frac{d}{dt}(y - x\theta) = V(y, x) - \theta U(y, x)$$

$$= (y - x\theta)\, W(y, x, \theta),$$

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provided the constant θ be determined so that

$$V(\theta, 1) = \theta U(\theta, 1).$$

Then $W(y, x, \theta) = c_0 y^{n-1} + c_1 y^{n-2} x + \ldots + c_{n-1} x^{n-1}$,

the coefficients c being given by the equations

$$c_r - \theta c_{r-1} = b_r - \theta a_r, \quad (r = 1, 2, \ldots, n-2),$$

with

$$c_0 = b_0 - \theta a_0,$$

$$-\theta c_{n-1} = b_n - \theta a_n.$$

The condition for the co-existence of these $n+1$ equations which determine the n quantities c is the foregoing equation

$$V(\theta, 1) = \theta U(\theta, 1),$$

which has $n+1$ roots, say $\theta_1, \theta_2, \ldots, \theta_{n+1}$.

For each of these roots we have

$$\frac{d}{dt}\{\log(y - x\theta_r)\} = W(y, x, \theta_r);$$

and therefore

$$\sum_{r=1}^{n+1} p_r \frac{d}{dt}\{\log(y - x\theta_r)\} = 0,$$

provided ratios of the quantities p_r can be found to satisfy the equations

$$\sum_{r=1}^{n+1} p_r c_s = 0, \quad (s = 0, 1, \ldots, n-1)$$

the value of c_s to be associated with p_r being that which depends upon θ_r.

From the equations which determine the quantities c we have

$$c_0 = b_0 - \theta a_0,$$

$$c_1 = b_0\theta + b_1 - \theta(\theta a_0 + a_1),$$

$$c_2 = b_0\theta^2 + b_1\theta + b_2 - \theta(\theta^2 a_0 + a_1\theta + a_2),$$

and so on. Writing

$$p_r = q_r U(\theta_r, 1),$$

we have

$$p_r c_s = q_r\{b_0\theta_r^s + b_1\theta_r^{s-1} + \ldots + b_s - \theta_r(a_0\theta_r^s + a_1\theta_r^{s-1} + \ldots + a_s)\} U(\theta_r, 1)$$

$$= q_r[(b_0\theta_r^s + \ldots + b_s) U(\theta_r, 1) - (a_0\theta_r^s + \ldots + a_s) V(\theta_r, 1)].$$

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This coefficient of q_r is one of the combinations to be made in forming the eliminant of $U(\theta, 1)$ and $V(\theta, 1)$ according to Bezout's process; writing it in the form

$$(s, 1)\,\theta_r^{n-1} + (s, 2)\,\theta_r^{n-2} + \ldots + (s, n),$$

we have in place of the equation

$$\sum_{r=1}^{n+1} p_r c_s = 0$$

the equation

$$(s, 1)\sum_{r=1}^{n+1}(q_r\theta_r^{n-1}) + (s, 2)\sum_{r=1}^{n+1}(q_r\theta_r^{n-2}) + \ldots + (s, n)\sum_{r=1}^{n+1} q_r = 0$$

for values $0, 1, 2, \ldots, n-1$ of s.

These n equations in the n quantities $\sum_{r=1}^{n+1}(q_r\theta_r^{n-p})$ can be satisfied only by

(1) the determinant of the coefficients vanishing, or by

(2) the vanishing of all the quantities $\sum_{r=1}^{n+1}(q_r\theta_r^{n-p})$.

Now the former is not possible; for it would imply that the determinant formed by constituents (s, p) vanishes, that is, that the resultant of $U(\theta, 1)$ and $V(\theta, 1)$ vanishes; and therefore that U and V have a common factor, contrary to the initial hypothesis. Hence we have

$$\sum_{r=1}^{n+1}(q_r\theta_r^{n-p}) = 0,$$

for $p = 1, 2, \ldots, n$; and therefore q_r is proportional to the difference-product

$$\zeta_r^{\frac{1}{2}}(\theta_{r+1}, \theta_{r+2}, \ldots, \theta_{r-1}), \text{ say } \zeta_r^{\frac{1}{2}},$$

taken in cyclical order for the $n+1$ quantities θ, the symbol θ_r being omitted from $\zeta_r^{\frac{1}{2}}$. Hence we may take

$$p_r = U(\theta_r, 1)\,\zeta_r^{\frac{1}{2}},$$

and therefore $\sum_{r=1}^{n+1} U(\theta_r, 1)\,\zeta_r^{\frac{1}{2}}\frac{d}{dt}\{\log(y - x\theta_r)\} = 0,$

the integral of which may be taken in the form

$$\prod_{r=1}^{n+1}\{(y - x\theta_r)^{U(\theta_r, 1)\,\zeta_r^{\frac{1}{2}}}\} = A,$$

where A is an arbitrary constant.

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This form of the integral may be shown to agree with the form derived by the ordinary method of integration applied to the equation considered as homogeneous.

For example, the primitive of

$$\frac{dy}{dx} = \frac{b_0y^2 + b_1yx + b_2x^2}{a_0y^2 + a_1yx + a_2x^2},$$

is

$$(y - x\theta_1)^{(\theta_2 - \theta_3)(a_0\theta_1^2 + a_1\theta_1 + a_2)} \times (y - x\theta_2)^{(\theta_3 - \theta_1)(a_0\theta_2^2 + a_1\theta_2 + a_2)} \times (y - x\theta_3)^{(\theta_1 - \theta_2)(a_0\theta_3^2 + a_1\theta_3 + a_2)} = A,$$

where A is an arbitrary constant and θ_1, θ_2, θ_3 are the roots of the equation

$$b_0\theta^2 + b_1\theta + b_2 = \theta(a_0\theta^2 + a_1\theta + a_2);$$

and similarly for other cases.

THE TRANSFORMATION OF MULTIPLE SURFACE INTEGRALS INTO MULTIPLE LINE INTEGRALS.

By *Alex. Mc Aulay*, Ormond College, Melbourne.

In the last volume of the *Messenger of Mathematics*, p. 23, Mr. Larmor has an article under this heading. The object of the present paper is to combat two of his chief conclusions and give others in their stead.

On p. 26 he says, first, that

$$\iint ds\,ds'\,f(r)\cos\varepsilon \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1),$$

and

$$\iint ds\,ds'\,\phi(r)\cos\vartheta\cos\vartheta' \quad \ldots\ldots\ldots\ldots\ldots\ldots (2),$$

are the most general double line integrals which treat space isotropically. Here ε, ϑ and ϑ' are the angles between ds and ds', ds and r, and ds' and r, respectively; and f, ϕ are any functions of r. He says, secondly, that it is clear that the second integral vanishes if either circuit be closed. I think both of these statements will be found to be untrue.

In fact, I propose to shew that the most general double line integrals in question are included under the above and a third form, viz.

$$\iint ds\,ds'\,F(r)\sqrt{(1 - \cos^2\varepsilon - \cos^2\vartheta - \cos^2\vartheta' + 2\cos\varepsilon\cos\vartheta\cos\vartheta')} = \iint ds\,ds'\,F(r)\sin\varepsilon\sin\varepsilon' \quad \ldots (3),$$

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where ε' is the angle between the plane of ds and ds' and the direction of r. Further, when both circuits are closed, I propose to show that the line integrals (1), (2), (3) become respectively the surface integrals

$$\iiiint dS dS' \left[r \frac{d}{dr} \left\{ \frac{1}{r} f'(r) \right\} \cos\theta \cos\theta' - \frac{1}{r} \frac{d}{dr} \{ r f'(r) \} \cos\eta \right] \quad \ldots(4)$$

(Mr. Larmor's result, eq. (12), p. 28),

$$\iiiint dS dS' \left[-r \frac{d}{dr} \left\{ \frac{1}{r^3} \phi(r) \right\} \cos\theta \cos\theta' + \frac{1}{r} \frac{d\phi}{dr} \cos\eta \right] \quad \ldots(5),$$

and

$$\left. \begin{aligned} & -\iiiint dS dS' \frac{d}{dr} \left[\frac{1}{r^2} \frac{d}{dr} \{ r^2 F(r) \} \right] \\ & \quad \times \sqrt{(1 - \cos^2\eta - \cos^2\theta - \cos^2\theta' + 2\cos\eta \cos\theta \cos\theta')} \\ & = -\iiiint dS dS' \frac{d}{dr} \left[\frac{1}{r^3} \frac{d}{dr} \{ r^3 F(r) \} \right] \sin\eta \sin\eta' \end{aligned} \right\} \quad \ldots(6).$$

Here θ, θ', η, η' bear exactly the same relations to the directions of the normals as did ϑ, ϑ', ε, ε' to the directions of the tangents. It is necessary to say, with regard to the sign of the integral (6), that ϵ' and η' are so chosen that the directions of r, ds and ds' form a right-handed system as also do r, normal of dS and normal of dS'.

Before proceeding to the investigations here indicated, I will show that, for a particular case, integral (2) does not vanish under the conditions mentioned by Mr. Larmor. Let the (closed) circuit whose element is ds be a semicircle, and let the line ds' be a single element situated at a point on the axis of the circle, the direction of ds' being parallel to the diameter bounding the semicircle. The curved part of this contour contributes nothing to integral (2), for at every point $\vartheta = \frac{1}{2}\pi$. For every point on the diameter $\vartheta = \vartheta'$ and is only $= \frac{1}{2}\pi$ at the centre; therefore $\cos\vartheta \cos\vartheta'$ is positive for every element of the diameter except at the centre, where it vanishes. Hence, if ϕ is positive for all values of r (which, as a particular case, can of course be assumed), integral (2) does not vanish.

It is to be observed that the surface integral (5) does not involve the second differential coefficient of ϕ. This probably led to Mr. Larmor's error.

I have given these results at the outset in Mr. Larmor's notation for comparison, but I propose to use quaternions.

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Let $d\rho$, $d\rho'$ be vector elements of any two curves, and let ρ be the vector from the one to the other. Let ∇ have the usual meaning of vector differentiation with regard to ρ. It is necessary of course to specify which end of ρ is to be considered variable. If ∇ operate twice, as will always be the case in this paper, it does not matter which end be chosen.

Let $Q(d\rho, d\rho')$ be any quaternion function of ρ, $d\rho$ and $d\rho'$, which is linear in each of the latter. Then the most general type of quaternion double line integral is

$$\iint Q(d\rho, d\rho').$$

It is easiest to transform to a surface integral in this general form; though of course it will be necessary for our purpose to find Q in a more particular form and see what the surface integral reduces to in that case.

On p. 27, Vol. XIV., of the *Messenger of Mathematics*, equation (I), I showed how to transform a single line integral such as we are now considering into a surface integral. We have here only to repeat the process twice. Using the notation of that paper, except that for $Uvds$ we now put the less cumbrous symbol dv, we see that

$$\iint Q(d\rho, d\rho') = -\iiint\iint Q(V dv \nabla', V dv' \nabla') \ldots\ldots\ldots (7).$$

The reason for the minus sign in the second member is that, if we use the plus sign as it appears in the process of transformation, we must suppose that for one of the ∇'s, one end of ρ is variable, and for the other, the other end; whereas, by merely changing the sign, we may suppose the differentiations to refer to the same end—which, it matters not.

We will first transform the integrals (1), (2), (3) into surface integrals and afterwards shew that these are the most general forms for the case specified. In our present notation these integrals become respectively

$$-\iint f S d\rho\, d\rho' \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (8),$$

$$\iint \phi\, SU\rho\, d\rho\, SU\rho\, d\rho' \ldots\ldots\ldots\ldots\ldots\ldots (9),$$

$$-\iint F SU\rho\, d\rho\, d\rho' \ldots\ldots\ldots\ldots\ldots\ldots (10),$$

f, ϕ, and F being any scalar functions $\tau\rho$, or r, as we shall still continue to call it.

In the following transformations of the ∇'s into differentiations with regard to r it may be as well to say, for the benefit of those who are not very familiar with quaternions, that every such transformation of ∇ as occurs below comes

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under one of two heads. First, if ∇ operate on a *scalar function of* r,

$$\nabla = U\rho \frac{d}{dr}.$$

Secondly, if ∇ operate on $S\alpha\rho$, α being a vector which is for the instant considered constant,

$$\nabla S\alpha\rho = -\alpha.$$

From equation (7) we see that

$$\iiint f S d\rho\, d\rho' = -\iiiint f S V dv \nabla' V dv' \nabla'.$$

Now

$$f S V dv \nabla' V dv' \nabla' = -S dv dv' \nabla'^2 f + S dv \nabla S dv' \nabla f$$

$$= \left(\frac{2}{r}\frac{df}{dr} + \frac{d^2 f}{dr^2}\right) S dv dv' + S dv \nabla \left(\frac{1}{r}\frac{df}{dr} S\rho\, dv'\right)$$

$$= \left(\frac{2}{r}\frac{df}{dr} + \frac{d^2 f}{dr^2}\right) S dv dv' - \frac{1}{r}\frac{df}{dr} S dv dv'$$

$$+ r\frac{d}{dr}\left(\frac{1}{r}\frac{df}{dr}\right) SU\rho\, dv\, SU\rho\, dv'$$

$$= \frac{1}{r}\frac{d}{dr}\left(r\frac{df}{dr}\right) S dv dv' + r\frac{d}{dr}\left(\frac{1}{r}\frac{df}{dr}\right) SU\rho\, dv\, SU\rho\, dv';$$

therefore

$$\iiint f S d\rho\, d\rho' = -\iiiint \left\{ r\frac{d}{dr}\left(\frac{1}{r}\frac{df}{dr}\right) SU\rho\, dv\, SU\rho\, dv' \right.$$

$$\left. + \frac{1}{r}\frac{d}{dr}\left(r\frac{df}{dr}\right) S dv dv' \right\} \ldots (11),$$

which expresses the equality of integrals (1) and (4).

Again, by equation (7),

$$\iint \phi\, SU\rho\, d\rho\, SU\rho\, d\rho' = -\iiiint \phi\, SU\rho\, dv \nabla' SU\rho\, dv' \nabla'.$$

To effect the differentiations here indicated it is convenient to split the second ∇' up into $\nabla_1 + \nabla_2$, as explained in the paper already referred to (*Messenger of Mathematics*, Vol. XIV., p. 26), thus

$$\phi\, SU\rho\, dv \nabla' SU\rho\, dv' \nabla' = \left(\frac{\phi}{r^3}\right)_1 S\rho_1 dv \nabla' S\rho_2 dv' (\nabla_1 + \nabla_2)$$

$$= S\rho\, dv \nabla' S\rho\, dv' \nabla \frac{\phi}{r^3} - \frac{\phi}{r^3} S dv \nabla' V\rho\, dv'$$

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[the first of these terms vanishes, since $\nabla \frac{\phi}{r^2}$ is parallel to ρ]

$$= \frac{\phi}{r^2}(S d\nu' \nabla' S d\nu\rho - S\rho \nabla' S d\nu d\nu')$$

$$= \frac{2\phi}{r^2} S d\nu d\nu' + r\frac{d}{dr}\left(\frac{\phi}{r^2}\right)(S d\nu d\nu' + SU\rho d\nu SU\rho d\nu')$$

$$= \frac{1}{r}\frac{d\phi}{dr} S d\nu d\nu' + r\frac{d}{dr}\left(\frac{\phi}{r^2}\right) SU\rho d\nu SU\rho d\nu';$$

therefore

$$\iint \phi SU\rho d\rho SU\rho d\rho' = -\iiiint \left\{ r\frac{d}{dr}\left(\frac{\phi}{r^2}\right) SU\rho d\nu SU\rho d\nu' + \frac{1}{r}\frac{d\phi}{dr} S d\nu d\nu' \right\} \ldots (12),$$

which expresses the equality of integrals (2) and (5).

Again, by equation (7),

$$\iint F SU\rho d\rho d\rho' = -\iiiint F SU\rho V d\nu \nabla' V d\nu' \nabla'.$$

Now

$$F SU\rho V d\nu \nabla' V d\nu' \nabla' = -F SU\rho \nabla' S d\nu d\nu' \nabla'$$

$$= S d\nu d\nu' \nabla \left(\frac{2F}{r} + \frac{dF}{dr}\right)$$

$$= \frac{d}{dr}\left\{\frac{1}{r^3}\frac{d}{dr}(r^3 F)\right\} SU\rho d\nu d\nu';$$

therefore

$$\iint F SU\rho d\rho d\rho' = -\iiiint \frac{d}{dr}\left\{\frac{1}{r^3}\frac{d}{dr}(r^3 F)\right\} SU\rho d\nu d\nu' \ldots (13),$$

which expresses the equality of integrals (3) and (6).

We have now to show that the functions we have been considering are the most general of the kind. Suppose the vector ρ and the two elements $d\rho$ and $d\rho'$ at its ends to form a rigid body. Since we consider space isotropic, the scalar functions we consider can only be *functions of the shape* of this rigid body. Let α and β be the vector components of $d\rho$, parallel and perpendicular to ρ, and let α' and β' be the similar components of $d\rho'$. Let γ and γ' be the vector components of β' parallel and perpendicular to β. Thus the shape of the rigid body is completely defined by $r, \tau\alpha, \tau\beta,$

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$\tau\alpha'$, $\tau\gamma$, $\tau\gamma'$. Since the functions in question are to be linear in both $d\rho$ and $d\rho'$, the most general forms must consist of a sum of quantities, each of which is an arbitrary function of r multiplied by one of the pair $\tau\alpha$ and $\tau\beta$ and by one of the three $\tau\alpha'$, $\tau\gamma$, and $\tau\gamma'$. But, further, the three products $\tau\alpha\tau\gamma$, $\tau\alpha\tau\gamma'$, and $\tau\alpha'\tau\beta$ cannot occur; for suppose $d\rho$ reduces to α and $d\rho'$ to γ, then our function reduces to a multiple of $\tau\alpha\tau\gamma$. Now change the sign of $d\rho'$. Since the function is linear in $d\rho'$, this changes the sign of the function; but the shape of the rigid body is not thereby altered, for it is merely turned through two right angles round α. Thus, the function both changes sign and is unaltered, *i.e.* it is zero. Similarly for the other two products mentioned. Thus the most general function which will satisfy all conditions is

$$l\tau\alpha\tau\alpha' + m\tau\beta\tau\gamma + n\tau\beta\tau\gamma',$$

where l, m, and n are arbitrary functions of r. Now

$$\tau\alpha\tau\alpha' = SU\rho\, d\rho\, SU\rho\, d\rho',$$

$$\tau\beta\tau\gamma = -S\beta\beta' = -S(U\rho\, V U\rho\, d\rho . U\rho\, V U\rho\, d\rho')$$

(by substituting the values of β and β' in terms of $d\rho$ and $d\rho'$)

$$= -Sd\rho\, d\rho' - SU\rho\, d\rho\, SU\rho\, d\rho'.$$

Similarly

$$\tau\beta\tau\gamma' = \tau V\beta\beta' = -SU\rho\beta\beta' = -S(U\rho\, U\rho\, V U\rho\, d\rho . U\rho\, V U\rho\, d\rho')$$

$$= -SU\rho\, d\rho\, d\rho'.$$

Thus our most general function becomes

$$(l-m)\, SV\rho\, d\rho\, SU\rho\, d\rho' - m\, Sd\rho\, d\rho' - n\, SU\rho\, d\rho\, d\rho',$$

which shows that all possible functions of the kind are included under the three forms we have already considered.

May 24, 1888.

NOTE BY MR. LARMOR ON THE PRECEDING PAPER.

Mr. Mc Aulay is correct in claiming that the two statements in my note referred to, which he says are in his view two of its chief conclusions, are erroneous.

The integral (2) was hastily put equal to zero, without any ground. Its true value, as here given, admits of simple

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verification by the method of my note, which after all is not remotely different from the quaternion analysis.

The other integral (3) failed to present itself owing to an implied restriction to rational linear functions of the angles between the directions of the lines. The formula

$$\sin\varepsilon\sin\varepsilon' = \frac{1}{r}\begin{vmatrix} x'-x, & y'-y, & z'-z \\ \frac{dx}{ds}, & \frac{dy}{ds}, & \frac{dz}{ds} \\ \frac{dx'}{ds'}, & \frac{dy'}{ds'}, & \frac{dz'}{ds'} \end{vmatrix},$$

however shows that the method of transformation applies to it.

It is no doubt merely by a slip of the pen, not unnatural in dealing with these delicate considerations, that Mr. Mc Aulay asserts that the three integrals here given are the most general double line integrals which treat space isotropically. The correct description occurs near the end, in conformity with the ideas contained in the note which he is correcting.

I may remark that the theorems are not true even in Mr. Mc Aulay's form when the circuits are linked together; they then require important modifications.

It is gratifying to find that my note has attracted such competent attention; and that the paragraph in question, though very faulty, has given rise to this elegant and interesting investigation.

A THEOREM IN ALGEBRA.

By *H. G. Dawson*, Christ's College, Cambridge.

If U be a binary form of the n^{th} degree, we easily see that

$$\frac{1}{U}\frac{dU}{dx} = \Sigma\frac{1}{(x-\alpha_1 y)},$$

$$\frac{1}{U}\frac{dU}{dy} = -\Sigma\frac{\alpha_1}{x-\alpha_1 y},$$

where α_1, &c., are the roots of $U=0$.

Hence $$\frac{1}{U}\left(p\frac{dU}{dx}+q\frac{dU}{dy}\right) = \Sigma\left(\frac{p-q\alpha_1}{x-y\alpha_1}\right).$$

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Again, proceeding to the second differentials,

$$\frac{1}{U}\frac{d^2U}{dx^2}=2\Sigma\frac{1}{(x-\alpha_1 y)(x-\alpha_2 y)},$$

$$\frac{1}{U}\frac{d^2U}{dx\,dy}=-\Sigma\frac{\alpha_1+\alpha_2}{(x-\alpha_1 y)(x-\alpha_2 y)},$$

$$\frac{1}{U}\frac{d^2U}{dy^2}=2\Sigma\frac{\alpha_1\alpha_2}{(x-\alpha_1 y)(x-\alpha_2 y)},$$

and we see that

$$\frac{1}{U}\left\{p^2\frac{d^2U}{dy^2}+2pq\frac{d^2U}{dx\,dy}+q^2\frac{d^2U}{dy^2}\right\}=2\Sigma\frac{(p-q\alpha_1)(p-q\alpha_2)}{(x-y\alpha_1)(x-y\alpha_2)},$$

Once more, the third differentials give

$$\frac{1}{U}\frac{d^3U}{dx^3}=6\Sigma\frac{1}{(x-\alpha_1 y)(x-\alpha_2 y)(x-\alpha_3 y)},$$

$$\frac{1}{U}\frac{d^3U}{dx^2\,dy}=-2\Sigma\frac{\alpha_1+\alpha_2+\alpha_3}{(x-\alpha_1 y)(x-\alpha_2 y)(x-\alpha_3 y)},$$

$$\frac{1}{U}\frac{d^3U}{dx\,dy^2}=2\Sigma\frac{\alpha_1\alpha_2+\alpha_2\alpha_3+\alpha_3\alpha_1}{(x-\alpha_1 y)(x-\alpha_2 y)(x-\alpha_3 y)},$$

$$\frac{1}{U}\frac{d^3U}{dy^3}=-6\Sigma\frac{\alpha_1\alpha_2\alpha_3}{(x-\alpha_1 y)(x-\alpha_2 y)(x-\alpha_3 y)},$$

shewing that

$$\frac{1}{U}\left(p\frac{d}{dx}+q\frac{d}{dy}\right)^3U=1.2.3.\Sigma\frac{(p-q\alpha_1)(p-q\alpha_2)(p-q\alpha_3)}{(x-\alpha_1 y)(x-\alpha_2 y)(x-\alpha_3 y)}.$$

The general theorem is then suggested that if

$$\Delta=p\frac{d}{dx}+q\frac{d}{dy},$$

where p, q are arbitrary and not operated on, then

$$\Delta^r U=r!\,U\left\{\Sigma\frac{(p-q\alpha_1)\ldots(p-q\alpha_r)}{(x-\alpha_1 y)\ldots(x-\alpha_r y)}\right\},$$

In order to prove this we will suppose the following equations to be true for m, and thence prove that they are true for $m+1$.

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The equations are

$$\frac{1}{U}\frac{d^m U}{dx^m} = (-1)^m m! \Sigma \frac{1}{(x-x_1)(x-x_2)\dots(x-x_m)},$$

$$\frac{1}{U}\frac{d^{m+1} U}{dx^m dy} = (-1)^{m+1}(m+1)! \Sigma \frac{1}{(x-x_1)(x-x_2)\dots(x-x_{m+1})},$$

$$\frac{1}{U}\frac{d^{m+2} U}{dx^m dy^2} = (-1)^{m+2}(m+2)! \Sigma \frac{1}{(x-x_1)(x-x_2)\dots(x-x_{m+2})},$$

$$\dots \qquad \dots$$

where $\Sigma f(x_1 \dots x_m)$ denotes the sum of the products m at a time of the n quantities $x_1 \dots x_n$.

Now

$$\frac{1}{U}\frac{d^{m+1} U}{dx^m dy} - \frac{1}{U^2}\frac{dU}{dx}\cdot\frac{d^m U}{dx^m dy}$$

$$= (-1)^{m+1}(m+1)! \Sigma \frac{-S(x_1 \dots x_m)}{(x-x_1)(x-x_2)\dots(x-x_m)(x-x_{m+1})},$$

therefore

$$\frac{1}{U}\frac{d^{m+1} U}{dx^{m+1-r} dy^r} = (-1)^r (m+1-r)! r! \Sigma \frac{1}{(x-x_1)(x-x_2)} \Sigma \frac{S(x_1 \dots x_m)}{(x-x_1)(x-x_2)\dots(x-x_m)}$$

$$- (-1)^r (m-r)! r! \Sigma \frac{S(x_1 \dots x_m)}{(x-x_1)(x-x_2)\dots(x-x_m)(x-x_{m+1})}$$

$$= (-1)^r (m-r)! r! \Sigma \frac{S(x_1 \dots x_m) + S(x_2 \dots x_{m+1}) + S(x_1 x_3 \dots x_{m+1})}{(x-x_1)(x-x_2)\dots(x-x_m)(x-x_{m+1})}$$

$$= (m+1-r)! r! (-1)^r \Sigma \frac{S_r(x_1 \dots x_{m+1})}{(x-x_1)\dots(x-x_{m+1})}.$$

Again, $\dfrac{1}{U}\dfrac{d^m U}{dy^m} = (-1)^m m! \Sigma \dfrac{x_1 x_2 \dots x_m}{(x-x_1)(x-x_2)\dots}$.

Differentiate with respect to y,

$$\frac{1}{U}\frac{d^{m+1} U}{dy^{m+1}} - \frac{1}{U^2}\frac{dU}{dy}\cdot\frac{d^m U}{dy^m}$$

$$= m!(-1)^m \Sigma \frac{x_1^2 x_2 \dots x_m}{(x-x_1)^2 (x-x_2)\dots(x-x_m)},$$

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therefore

$$\frac{1}{U}\frac{d^{m+1}U}{dy^{m+1}} = (-1)^{m-1}\,m!\,\Sigma\frac{\alpha_1}{x-\alpha_1 y}\,\Sigma\frac{\alpha_1\alpha_2\ldots\alpha_m}{(x-\alpha_1 y)\ldots(x-\alpha_m y)}$$

$$+(-1)^m\,m!\,\Sigma\frac{\alpha_1^2\alpha_2\ldots\alpha_m}{(x-\alpha_1 y)^2(x-\alpha_1 y)\ldots(x-\alpha_m y)}$$

$$=(-1)^{m+1}\,m!\,\Sigma\frac{(m+1)\,\alpha_1\ldots\alpha_{m+1}}{(x-\alpha_1 y)\ldots(x-\alpha_{m+1}y)}.$$

Hence the set of equations if true for m is true for $m+1$; and therefore being true for 2, 3, is universally true.

Multiplying by p^m, $p^{m-1}q$, &c., and adding, we get

$$\Delta^m U = m!\,U\left\{\Sigma\frac{(p-\alpha_1 q)\ldots(p-\alpha_m q)}{(x-\alpha_1 y)\ldots(x-\alpha_m y)}\right\}.$$

We may deduce many interesting results.

(1) If $p=x$ and $q=y$, we have

$$\left(x\frac{d}{dx}+y\frac{d}{dy}\right)^m U = m!\,U\left\{\frac{n.n-1\ldots(n-m+1)}{m!}\right\}$$

$$=n.n-1\ldots(n-m+1)\,U,$$

which is Euler's theorem.

(2) If $p=y$, $q=x$, then

$$\left(y\frac{d}{dx}+x\frac{d}{dy}\right)^m U = m!\,U\,\Sigma\frac{(y-\alpha_1 x)\ldots(y-\alpha_m x)}{(x-\alpha_1 y)\ldots(x-\alpha_m y)}.$$

(3) $\left(x'\frac{d}{dx}+y'\frac{d}{dy}\right)^m$, U being the symbolic representation of the mth emanant of U, it follows that the mth emanant expressed in terms of the roots is

$$\Sigma(x'-\alpha_1 y')(x'-\alpha_2 y')\ldots(x'-\alpha_m y')(x-\alpha_{m+1}y)(x-\alpha_{m+2}y)\ldots(x-\alpha_n y).$$

Christ's College,
Nov. 1st, 1888.

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ANALYTICAL FORMULÆ IN REGARD TO AN OCTAD OF POINTS.

By *Professor Cayley*.

THE term "tetrad" is used in two distinct but not inconsistent senses, viz., a tetrad denotes any four points; and it also denotes the four vertices of a self-conjugate tetrahedron in regard to a quadric surface. In fact, given any four points, there exists a triply infinite series of quadric surfaces such that in regard to any one of them the four points form a self-conjugate tetrahedron.

Two or more tetrads, in regard to one and the same quadric surface, are called similar tetrads.

The eight points of intersection of any three quadric surfaces are an octad; and we have the theorem that any seven points determine the octad, viz. the quadric surfaces which pass through any seven given points pass also through a uniquely determinate eighth point.

We have the further theorem that any two similar tetrads form an octad.

In particular the vertices of the tetrahedron

$$(x=0,\ y=0,\ z=0,\ w=0),$$

or say the points 1000, 0100, 0010, 0001 are a tetrad in regard to the quadric surface $x^2+y^2+z^2+w^2=0$. The points (x_1, y_1, z_1, w), (x_2, y_2, z_2, w), (x_3, y_3, z_3, w_3), (x_4, y_4, z_4, w_4), or say the points 1, 2, 3, 4, will be a tetrad in regard to the same quadric surface, if only

$$x_1x_2+y_1y_2+z_1z_2+w_1w_2=0, \text{ \&c. (six equations).}$$

Hence, these equations being satisfied, the two tetrads form an octad; the equation of a general quadric surface through the points of the first tetrad is

$$fyz+gzx+hxy+lxw+myw+nzw=0,$$

and we can from three such equations eliminate any two terms; the two terms eliminated may be such as xy, xz (12 forms) or such as yz, xw (3 forms); and the equation may thus be presented either in the form

$$fyz+(lx+my+nz)\,w=0,$$

or in the form

$$gzx+hxy+myw+nzw=0.$$

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The absolute magnitudes of the coordinates

$$(x_1, y_1, z_1, w_1), \text{ \&c.}$$

of these points are properly indeterminate; but we may if we please fix the absolute magnitudes by assuming

$$x_1^2 + y_1^2 + z_1^2 + w_1^2 = M, \text{ \&c.}$$

(four equations, the same quantity M in each of them), and we have as before

$$x_1x_2 + y_1y_2 + z_1z_2 + w_1w_2 = 0, \text{ \&c. (six equations)},$$

viz. the coordinates are taken to satisfy these 10 equations; and this being so, it is to be shown that there exist quadric surfaces as above. The 10 equations may be expressed in a correlative form in like manner with the ordinary six equations for the transformation of the rectangular coordinates (x, y, z), viz. the new form is

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 = M, \text{ \&c. (four equations)},$$

and $$x_1y_1 + x_2y_2 + x_3y_3 + x_4y_4 = 0, \text{ \&c. (six equations)},$$

and the coordinates thus also satisfy these 10 equations.

Thus the equation $(fyz + lx + my + nz)w = 0$, if we determine the coefficients in such wise that the surface passes through the points 1, 2, 3, becomes

$$\begin{vmatrix} yz, & xw, & yw, & zw \\ y_1z_1, & x_1w_1, & y_1w_1, & z_1w_1 \\ y_2z_2, & x_2w_2, & y_2w_2, & z_2w_2 \\ y_3z_3, & x_3w_3, & y_3w_3, & z_3w_3 \end{vmatrix} = 0,$$

and this equation must therefore be satisfied on writing therein (x_4, y_4, z_4, w_4) for (x, y, z, w); viz. the equation thus obtained, det. $(1, 2, 3, 4) = 0$, must be satisfied in virtue of the equations which connect the coordinates of the four points. But, instead of effecting this verification, it is better to exhibit the equation in (x, y, z, w) in a form wherein the coordinates of the four points enter symmetrically; and the new equation *quà* transformation of the original form is satisfied for $(x, y, z, w) = (x_1, y_1, z_1, w_1)$; and then of course, by reason of the symmetry, it is also satisfied for

$$(x, y, z, w) = (x_4, y_4, z_4, w_4).$$

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The transformation may be effected directly; but I prefer to write down the final result, and afterwards verify it.

The form is

$$yz \cdot Mw_1w_2w_3w_4$$

$$+ xw\,[- x_1y_1z_1 \cdot w_2w_3w_4 - x_2y_2z_2 \cdot w_3w_4w_1 - x_3y_3z_3 \cdot w_4w_1w_2 - x_4y_4z_4 \cdot w_1w_2w_3]$$

$$+ yw\,[\; x_1^2w_1 \cdot z_2z_3z_4 + x_2^2w_2 \cdot z_3z_4z_1 + x_3^2w_3 \cdot z_4z_1z_2 + x_4^2w_4 \cdot z_1z_2z_3]$$

$$+ zw\,[\; x_1^2w_1 \cdot y_2y_3y_4 + x_2^2w_2 \cdot y_3y_4y_1 + x_3^2w_3 \cdot y_4y_1y_2 + x_4^2w_4 \cdot y_1y_2y_3] = 0.$$

We have to show that the equation is satisfied by writing therein $(x, y, z, w) = (x_1, y_1, z_1, w_1)$. Calling the resulting value Ω, we have

$$\Omega_1 \div w_1 = My_1z_1w_2w_3w_4$$

$$- x_1^2y_1z_1w_2w_3w_4 - x_1(x_2y_2z_2w_3w_4w_1 + x_3y_3z_3w_4w_1w_2 + x_4y_4z_4w_1w_2w_3)$$

$$+ x_1^2w_1(y_1z_2z_3z_4 + z_1y_2y_3y_4)$$

$$+ y_1z_1[x_2^2w_2(z_3z_4 + y_3y_4) + x_3^2w_3(z_4z_2 + y_4y_2) + x_4^2w_4(z_2z_3 + y_2y_3)],$$

which is

$$= y_1z_1w_2w_3w_4(x_2^2 + x_3^2 + x_4^2) - x_1(x_2y_2z_2w_3w_4w_1 + x_3y_3z_3w_4w_1w_2 + x_4y_4z_4w_1w_2w_3)$$

$$+ x_1^2w_1(y_1z_2z_3z_4 + z_1y_2y_3y_4)$$

$$- y_1z_1[x_2^2w_2(x_3x_4 + w_3w_4) + x_3^2w_3(x_2x_4 + w_2w_4) + x_4^2w_4(x_2x_3 + w_2w_3)].$$

The expression contains terms in $w_2w_3w_4$ which destroy each other, and omitting them we have

$$\Omega_1 \div w_1 = - x_1(x_2y_2z_2w_3w_4w_1 + x_3y_3z_3w_4w_1w_2 + x_4y_4z_4w_1w_2w_3) + x_1^2w_1(y_1z_2z_3z_4 + z_1y_2y_3y_4) - x_2x_3x_4y_1z_1(x_2w_2 + x_3w_3 + x_4w_4),$$

where the last line is $= x_2x_3x_4y_1z_1 \cdot x_1w_1$,

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Hence the whole divides by x_1w_1, or we have

$$\Omega \div x_1w_1^2 = -x_2y_2z_2 . w_3w_4 - x_3y_3z_3 . w_4w_2 - x_4y_4z_4 . w_2w_3 + x_1y_1z_2z_3z_4 + x_1z_1y_2y_3y_4 + y_1z_1x_2x_3x_4,$$

viz. this is

$$\begin{aligned} = \; & x_2x_3x_4y_1z_1 + y_2y_3y_4z_1x_1 + z_2z_3z_4x_1y_1 \\ & + x_2y_2z_2(x_3x_4 + y_3y_4 + z_3z_4) \\ & + x_3y_3z_3(x_2x_4 + y_2y_4 + z_2z_4) \\ & + x_4y_4z_4(x_2x_3 + y_2y_3 + z_2z_3); \end{aligned}$$

or finally it is

$$\begin{aligned} = \; & x_2x_3x_4(y_1z_1 + y_2z_2 + y_3z_3 + y_4z_4) \\ & + y_2y_3y_4(z_1x_1 + z_2x_2 + z_3x_3 + z_4x_4) \\ & + z_2z_3z_4(x_1y_1 + x_2y_2 + x_3y_3 + x_4y_4), \end{aligned}$$

which is $= 0$; that is we have $\Omega_1 = 0$, the required result.

The foregoing equation of the quadric surface can be by mere interchanges of the letters exhibited in 12 different forms.

We may, in like manner, first establish and then verify the equation

$$\begin{aligned} & zx(-z_1^2w_1y_2y_3y_4 - z_2^2w_2y_3y_4y_1 - z_3^2w_3y_4y_1y_2 - z_4^2w_4y_1y_2y_3) \\ & +xy(\; y_1^2w_1z_2z_3z_4 + y_2^2w_2z_3z_4z_1 + y_3^2w_3z_4z_1z_2 + y_4^2w_4z_1z_2z_3) \\ & +yw(-x_1y_1^2z_2z_3z_4 - x_2y_2^2z_3z_4z_1 - x_3y_3^2z_4z_1z_2 - x_4y_4^2z_1z_2z_3) \\ & +zw(\; z_1^2x_1y_2y_3y_4 + z_2^2x_2y_3y_4y_1 + z_3^2x_3y_4y_1y_2 + z_4^2x_4y_1y_2y_3) = 0; \end{aligned}$$

this can be by a cyclical interchange of the letters (x, y, z) exhibited in 3 different forms.

Of course the fifteen equations belong each of them to a quadric surface through the 8 points. Any three of the fifteen equations, say $U=0$, $V=0$, $W=0$, may be used for determining the octad, and the equation of any other quadric through the octad is then $\alpha U + \beta V + \gamma W = 0$.

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NOTE ON CONJUGATE ANNIHILATORS OF HOMOGENEOUS AND ISOBARIC DIFFERENTIAL EXPRESSIONS.

By *L. J. Rogers.*

§ 1. SUPPOSE we have a class of homogeneous and differential expressions possessing some internal quality, geometric or otherwise, which may be tested externally by an annihilator, sufficient and necessary for the test, and of the first order in differential operations. Suppose also that new functions of the class can be formed by simple differentiation, that is by a generator of the form $\frac{1}{y_1}\frac{d}{dx}$. Using y_n to denote $\frac{d^n y}{dx^n}$, δ_n to denote the partial differentiation-symbol $\frac{d}{dy_n}$, and δ to denote $\frac{d}{dx}$, we shall assume the annihilator in question to have the form

$$A_1\delta_1 + A_2\delta_2 + A_3\delta_3 + \ldots,$$

which we shall write A for shortness.

We shall assume also that the order of A_n is not greater than $(n-1)$, *i.e.* that A_n is a function of $y_1, y_2 \ldots y_{n-1}$, and not of y_n. It is easy to shew that we are justified in this assumption. Suppose we have an annihilable function, say of y_1, y_2, y_3.

Let it be written

$$y_3^n + a_1 y_3^{n-1} + \ldots,$$

and suppose

$$A_3 = k_0 y_3^m + k_1 y_3^{m-1} + \ldots k_m,$$

and that $a_1, a_2 \ldots, A_2, k_0, k_1 \ldots k_m$ are functions of y_1, y_2.

Then using the annihilator, we have

$$(k_0 y_3^m + \ldots k_m)(n y_3^{n-1} + \ldots) + \frac{da_1}{dy_2} y_3^{n-1} + \ldots = 0,$$

i.e. identically zero.

Hence equating coefficients of powers of y_3, we get

$$k_0, k_1, \ldots k_{m-1} \text{ all } = 0.$$

that is A only contains y_1, y_2.

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It is of course useless to suppose that A_2 contains any higher suffixes. Under these circumstances it is obvious that A_1 must be zero, and A_2 must be a power of y_1.

It is moreover impossible to form an homogeneous and isobaric expression out of y_1 and y_2, otherwise than each of these quantities taken alone, so that the first function of the class in question will be of the form $y_1y_3 - \alpha y_2^2$.

Suppose then that the first "absolute" function of the class to be $\frac{y_1y_3 - \alpha y_2^2}{y_1^m}$, where by absolute, I mean, that we divide the zero-equand $y_1y_3 - \alpha y_2^2$ by such a power of y_1 that the generator $\frac{1}{y_1^s}\frac{d}{dx}$ operating on the fraction thus obtained will produce a new annihilable function; s being a power chosen so as to make the new function also absolute.

We derive in this way a set of absolute protomorphs

$$y_1y_3 - \alpha y_2^2, \quad y_1^2y^4 - \ldots, \quad y_1^3y_5 - \ldots,$$

where obviously the weight w = twice the degree θ.

Moreover
$$\left.\begin{aligned} \theta &= y_1\delta_1 + y_2\delta_2 + \ldots \\ w &= y_1\delta_1 + 2y_2\delta_2 + 3y_3\delta_3 + \ldots \end{aligned}\right\} \ldots\ldots\ldots\ldots (1).$$

It is interesting to note that these two grade-equations are the only two possible, of the first degree in the coefficients of the δ's.

In fact, suppose the operation

$$y_1\delta_1 + 3y_2\delta_2 + 6y_3\delta_3 + 10y_4\delta_4 + \ldots$$

were an operator equivalent to multiplying by a constant.

Then we could eliminate y_1, y_2 and get a similar operator $y_3\delta_3 + \ldots \equiv$ constant.

But operating on $y_1y_3 - \alpha y_2^2$ we should have

$$y_1y_3 = (y_1y_3 - \alpha y_2^2) \times \text{constant},$$

which is impossible. This process of proof is obviously general.

We now proceed to find the form of the annihilator A.

Let F_θ be an annihilable of degree θ, and absoluting index i.

Then also is $\frac{d}{dx}\frac{F_\theta}{y_1^i}$ annihilable,

i. e.
$$A\left(y_1\frac{d}{dx}F_\theta - iy_2F_\theta\right) = 0,$$

therefore
$$y_1A\frac{d}{dx}F_\theta - iF_\theta A_2 = 0.$$

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But $$A\frac{d}{dx}F_\theta = \left(A\frac{d}{dx} - \frac{d}{dx}A\right)F_\theta$$

$$= (A_2\delta_2 + A_3\delta_3 + \ldots)(y_2\delta_1 + y_3\delta_2 + \ldots) - (y_2\delta_1 + y_3\delta_2 + \ldots)(A_2\delta_2 + \ldots),$$

where differential operations of the second degree cancel, and leave only

$$A_2\delta_1 + (A_3 - \delta A_2)\,\delta_2 + (A_4 - \delta A_3)\,\delta_3 + \ldots\ldots\ldots(2),$$

therefore $$y_1\{A_2\delta_1 + (A_3 - \delta A_2)\,\delta_2 + \ldots\} = iF_\theta A_2.$$

Thus iF_θ must admit of being represented in the form

$$(y_1\delta_1 + ky_2\delta_2 + k'y_3\delta_3 + \ldots)F_\theta,$$

where k, k' are numerical.

But this can only be done by the operators for θ and w, as shewn above. Hence i must be a function of θ and w, *i.e.* a multiple of θ, since $w = 2\theta$; while the generator is $\frac{1}{y_1^k}\delta$.

Let $i = (k+1)\,\theta$.

Then we can easily shew that

$$A_2 iF_\theta = A_2\{y_1\delta_1 + (k+1)\,y_2\delta_2 + (2k+1)\,y_3\delta_3 + \ldots\}.$$

Equating this to (2) we get

$$y_1A_2 = y_1A_2,$$

$$y_1(A_3 - \delta A_2) = (k+1)\,y_2A_2,$$

$$A_4 - \delta A_3 = (2k+1)\,y_3A_2,$$

&c.,

and since $$A(y_1y_3 - \alpha y_2^2) = 0,$$

we get $$y_1A_3 - 2\alpha y_2A_2 = 0,$$

$$y_1\delta A_2 = (2\alpha - k - 1)\,y_2A_2,$$

whence $$A_2 = y_1^{2\alpha-k-1},$$

$$A_3 = 2\alpha \cdot y_1^{2\alpha-k-2}y_2,$$

&c.

In the case of homographic invariants, we have $2\alpha = 3$, $i = 2\theta$, $k = 1$, so that $A_2 = y_1$, $A_3 = 3y_2$, &c.

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Let us now find the conjugate annihilator, that is the operator which annihilates the equivalent functions expressed in forms of $x_1, x_2 \ldots$, where x_n denotes $\frac{d^n x}{dy^n}$.

We have $$\frac{y_1 y_3 - \alpha y_2^2}{y_1^{2(k+1)}} = -\frac{x_1 x_3 + (\alpha - 3) x_2^2}{x^{6-2(k+1)}},$$

as is easily shewn.

Calling the right side

$$-\frac{x_1 x_3 - \alpha' x_2^2}{x_1^{2(k'+1)}},$$

we get $$\alpha' + \alpha = 3,$$

$$2(k+1) + 2(k'+1) = 6 \text{ or } k + k' = 1,$$

therefore $$(2\alpha - k - 1) + (2\alpha' - k' - 1) = 3,$$

and the conjugate annihilator is

$$x_1^{2\alpha' - k' - 1} \frac{d}{dx_2} + 2\alpha' x_2 x_1^{2\alpha' - k' - 2} \frac{d}{dx_3} + \ldots.$$

To shew that we actually get the same functions in terms of x, we have merely to prove that the generators $\frac{1}{y_1^k} \frac{d}{dx}$ and $\frac{1}{x_1^{k'}} \frac{d}{dy}$ are equivalent. This is perfectly obvious since $k + k' = 1$.

We have then established in full the connection between such conjugate annihilator.

§ 2. These results may be written in a simpler form.

If $$y_1^m \frac{d}{dy_2} + \alpha y_1^{m-1} y_2 \frac{d}{dy_3} + A_4 \frac{d}{dy_4} + \ldots$$

be a test-annihilator for a class of homogeneous and isobaric functions of $y_1, y_2, y_3 \ldots$, then these functions when expressed in terms of $x_1, x_2, x_3 \ldots$ are annihilable by

$$x_1^m \frac{d}{dx_2} + b x^{n-1} x_2 \frac{d}{dx_3} + B_4 \frac{d}{dx_4} + \ldots,$$

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where
$$m+n=3,$$
$$a+b=6,$$
$$A_4-\delta A_3=(a-n)\,y_2y_1^{m-1},$$
$$A_5-\delta A_4=(2a-2n-1)\,y_2y_1^{m-1},$$
$$A_6-\delta A_5=(3a-3n-2)\,y_4y_1^{m-1},$$
$$\&c.,$$

with similar equations for determining B_4, B_5....

From this we see that in order that both annihilators should be integral, we can only have $m=1$, $n=2$ or $m=2$, $n=1$, *i.e.* one annihilator is lineo-linear while its conjugate is quadro-linear.

Thus for homographic invariants one annihilator is

$$y_1\delta_2+3y_2\delta_3+6y_3\delta_4+\ldots,$$

while its conjugate is

$$x_1^2\frac{d}{dx_2}+3x_1x_2\frac{d}{dx_3}+\ldots.$$

Homographic reciprocants have of course two annihilators.

The self-conjugate annihilator V for pure reciprocants is not included in our investigation, which only treats of functions including y_1.

§ 3. There is another class of mixed homogeneous reciprocants of some interest, although the geometric reciprocity of the variables is not very clearly defined.

These may be called Reduced Self-Polar Reciprocants, being obtained by reducing by unity all the suffixes in self-polar reciprocants.

If any pure reciprocant be so reduced, the result obtained has an annihilator similar to V, namely

$$3y_1^2\frac{d}{dy_2}+10y_1y_2\frac{d}{dy_3}+\ldots.$$

Here $m=2$, $a=\frac{10}{3}$, so that $n=1$, $b=\frac{8}{3}$, and we easily get

$$3x_1\frac{d}{dx_2}+8x_2\frac{d}{dx_3}+15x_3\frac{d}{dx_4}+\ldots,$$

which is similar to the reduced projective annihilator.

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Hence a reduced projective reciprocant has two mutually conjugate annihilators; but as I have shewn elsewhere in *London Math. Soc. Proceedings*, Vol. XVII., Nos. 274, 275, it is only self-polars which when reduced give rise to true homogeneous reciprocants. In fact no annihilators can alone be tests for mixed reciprocants, as is well known.

There are an infinity of similar pair of conjugate reciprocants, but the above considered are the only ones that have hitherto been shewn to have geometric interest.

Leeds,
Oct. 18*th*, 1888.

ON A CERTAIN MINIMUM THEOREM.

By *J. Brill, M.A.*, St. John's College.

1. In a paper in the fifty-fourth volume of *Crelle*, entitled "Ueber eine allgemeine Transformation der hydrodynamischen Gleichungen", Clebsch has given a minimum theorem relating to the steady motion of a perfect incompressible fluid; but he has not fully discussed the conditions under which this theorem holds. When we form the variation of an integral we have two sets of terms, viz. the terms at the limits and the terms under the integral sign. Now Clebsch has shewn that in the case he considers the hydrodynamical equations cause the terms under the integral sign to vanish, but he has not shewn what becomes of the terms at the limits. Further, the vanishing of the terms of the first order proves only that the value of the integral is stationary, and it is necessary to discuss the terms of the second order to discover whether the value of the integral is actually a maximum or a minimum. In the following paper I propose to give the interpretation of the vanishing of the terms at the limits, and far as possible to discuss the discrimination of cases.

2. The general form of the theorem will be more readily understood if in the first instance we consider the form that it takes in two important particular cases* which Clebsch shews to be deducible from his general theory. These are firstly the case of plane steady motion, and secondly the case of steady motion symmetrical with respect to an axis.

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* It will be seen later that we can work out the theory in these two cases more completely than in the general case.

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In the case of plane steady motion we have a stream function ψ, which satisfies an equation of the form

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + f(\psi) = 0.$$

The energy integral also takes the form

$$\frac{p}{\rho} + V + \tfrac{1}{2}q^2 + F(\psi) = \text{const.},$$

where $F(\psi) = \int f(\psi)\, d\psi$.

Consider now the expression

$$U = \iint \left[\tfrac{1}{2} \left\{ \left(\frac{\partial\psi}{\partial x}\right)^2 + \left(\frac{\partial\psi}{\partial y}\right)^2 \right\} - F(\psi) \right] dx\, dy.$$

Taking the variation of this integral, we obtain

$$\delta U = \iint \left\{ \frac{\partial\psi}{\partial x}\frac{\partial\delta\psi}{\partial x} + \frac{\partial\psi}{\partial y}\frac{\partial\delta\psi}{\partial y} - f(\psi)\,\delta\psi \right\} dx\, dy$$

$$= \int \delta\psi \frac{\partial\psi}{\partial n}\, ds - \iint \left\{ \frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + f(\psi) \right\} \delta\psi\, dx\, dy.$$

This follows by Green's Theorem, the line integral being taken round the bounding curves of the space throughout which the surface integral is taken, it being understood that $\partial\psi/\partial n$ denotes the rate of variation of ψ in the direction of the normal to the bounding curve drawn outwards from the fluid.

The double integral vanishes in virtue of the equation quoted above; and if we introduce the condition that the value of ψ shall be fixed along the boundary, the single integral vanishes also, and thus the value of U becomes stationary. Further, the condition that the value of ψ should be given along the boundary is tantamount to the condition that the normal velocity should be given along the boundary. For if the value of ψ be known along the boundary, and the form of the boundary be also known, then we can express ψ in terms of s, where s is an arc of the bounding curve measured from some fixed point. Thus we can calculate the value of $d\psi/ds$, which is the normal velocity. Similarly, if the normal velocity be given along the boundary we can express it in terms of s, and thence by integration obtain the value of ψ.

Thus our theorem may be stated as follows:

There are an indefinite number of values of the stream

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function that give a stated normal velocity at the boundary, but among these there is a certain one that satisfies the equation

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + f(\psi) = 0,$$

or, at all events, a certain definite number that satisfy this equation. This particular value, or each of these particular values, as the case may be, makes the value of the integral

$$\iint\left[\tfrac{1}{2}\left\{\left(\frac{\partial\psi}{\partial x}\right)^2 + \left(\frac{\partial\psi}{\partial y}\right)^2\right\} - F(\psi)\right] dx\,dy$$

stationary.

3. We now pass on to discuss, as far as is possible in the general case, the question as to whether the value of the said integral given by one of the above-mentioned values is actually a minimum. The terms of the second order in the variation of U are

$$\tfrac{1}{2}\iint\left\{\left(\frac{\partial\delta\psi}{\partial x}\right)^2 + \left(\frac{\partial\delta\psi}{\partial y}\right)^2 - f'(\psi)\,\delta\psi^2\right\} dx\,dy.$$

Hence, if $f'(\psi)$ be negative for all values of ψ, the terms of the second order are necessarily positive, and in this case the value of the integral will be a minimum.

An important case is that in which $f(\psi)$ is constant, and therefore $f'(\psi)$ equal to zero. This corresponds to the case of constant vorticity, which contains as a particular case the irrotational solution under the given conditions. Another important case is that in which $f(\psi) = -c^2\psi$. The condition will also be satisfied if we write

$$f(\psi) = -(a_0 + a_1\psi + a_3\psi^3 + a_5\psi^5 + \ldots),$$

where a_1, a_3, a_5, ... are positive constants, *e.g.*

$$f(\psi) = -c^2 \sinh \psi.$$

To discuss the case generally, we will write

$$W = \tfrac{1}{2}\left\{\left(\frac{\partial\psi}{\partial x}\right)^2 + \left(\frac{\partial\psi}{\partial y}\right)^2\right\} - F(\psi).$$

The criteria originally given by Brunacci in 1806 for deciding whether any particular solution gives rise to a maximum or minimum value of an integral of the form we

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are considering, are the following. The expression

$$\frac{\partial^2 W}{\left(\partial \frac{\partial \psi}{\partial x}\right)^2}$$

should be of invariable sign throughout the portion of space we are considering, and the expression

$$\frac{\partial^2 W}{\left(\partial \frac{\partial \psi}{\partial x}\right)^2} \frac{\partial^2 W}{\left(\partial \frac{\partial \psi}{\partial y}\right)^2} - \left\{\frac{\partial^2 W}{\partial \frac{\partial \psi}{\partial x} \partial \frac{\partial \psi}{\partial y}}\right\}^2$$

should be always positive, although it might vanish at some point (see Carll's *Calculus of Variations*, Art. 431). In the case under consideration each of these expressions becomes unity, and thus the above conditions are satisfied for every possible solution.

These criteria, however, are sufficient only in the case in which the space throughout which the integral is taken is small. Mr. Culverwell has recently shewn (*Phil. Trans.*, 1887) that in the case where the region of integration is of finite magnitude, we must have in addition the condition that there should be no other solution which gives rise to a maximum or a minimum value. If there be another solution which is of this character, then the solution under consideration will give rise to a minimax value. Thus it is clear that in any case there is one, and there can only be one, solution which gives rise to a maximum or minimum value, all other solutions giving rise to minimax values. Therefore in the cases for which we have shewn that all solutions that arise must necessarily give rise to minimum values, there can only be one solution. I do not consider the usual argument for uniqueness of solution from physical considerations to be conclusive. Of course in any given case there will only be one solution which is physically possible, but there may possibly be solutions which are analytically possible but physically impossible. In cases like the present it is possible that the one solution which is physically possible may coincide with the one solution that gives rise to a maximum or to a minimum value.

If we have a solution which gives rise to a maximum or to a minimum value, then it is necessary to examine it on its own merits to discover to which class it belongs.

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4. In Art. 2, we made use of Green's Theorem to effect the necessary transformation in the terms of the first order in the variation of U. If the space throughout which the integral is taken be multiply connected, then, in addition to the terms at the limits, we shall have a series of single integrals corresponding to the barriers that have to be drawn in order to render the region simply connected. Each of these integrals will be of the form

$$\delta\kappa \int \frac{\partial \psi}{\partial n}\, ds,$$

where κ denotes that cyclic constant of ψ which corresponds to the circuit across which the barrier is drawn. For these terms to disappear it will be necessary for each $\delta\kappa$ to vanish. Thus, in the case of a multiply connected region, it will be necessary to have given, in addition to the normal velocity at the boundary, the cyclic constants of ψ corresponding to the various circuits.

5. We now pass on to consider briefly the case of steady motion symmetrical with respect to a fixed axis. In this case we have Stokes's stream function, which obeys the equation

$$\frac{\partial^2 \psi}{\partial r^2} - \frac{1}{r}\frac{\partial \psi}{\partial r} + \frac{\partial^2 \psi}{\partial z^2} + r^2 f(\psi) = 0.$$

The energy integral also takes the form

$$\frac{p}{\rho} + V + \tfrac{1}{2}q^2 + F(\psi) = \text{const.},$$

where $F(\psi) = \int f(\psi)\, d\psi$, and the integral whose variation we have to consider will be

$$\iint \left[\frac{1}{2r} \left\{ \left(\frac{\partial \psi}{\partial r}\right)^2 + \left(\frac{\partial \psi}{\partial z}\right)^2 \right\} - rF(\psi) \right] dr\, dz.$$

Calling this integral U, we shall obtain

$$\delta U = \int \frac{\delta\psi}{r}\frac{\partial \psi}{\partial n}\, ds - \iint \left\{ \frac{\partial^2 \psi}{\partial r^2} - \frac{1}{r}\frac{\partial \psi}{\partial r} + \frac{\partial^2 \psi}{\partial z^2} + r^2 f(\psi) \right\} \frac{dr\, dz}{r}.$$

The double integral vanishes in virtue of the equation quoted above, and the single integral will also vanish if the value of ψ be fixed along the boundary. It may be easily shewn, as in the former case, that the condition that the value of ψ should be given along the boundary is tantamount to

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the condition that the normal velocity should be given along the boundary. Thus the statement of the theorem takes exactly the same form as in the former case, and exactly the same restriction applies when the region throughout which the integral is taken is multiply connected.

Further, the terms of the second order in the variation of U are

$$\tfrac{1}{2}\iint\frac{1}{r}\left\{\left(\frac{\partial\delta\psi}{\partial r}\right)^2+\left(\frac{\partial\delta\psi}{\partial z}\right)^2-r^2f'(\psi)\,\delta\psi^2\right\}dr\,dz.$$

Thus we see that in this case also any solution must necessarily give a minimum if $f'(\psi)$ be negative for all values of ψ, for r is essentially positive. And, calling the expression under the integral sign in the original integral W, we have

$$\frac{\partial^2 W}{\left(\partial\dfrac{\partial\psi}{\partial r}\right)^2}=\frac{1}{r},$$

and

$$\frac{\partial^2 W}{\left(\partial\dfrac{\partial\psi}{\partial r}\right)^2}\frac{\partial^2 W}{\left(\partial\dfrac{\partial\psi}{\partial z}\right)^2}-\left\{\frac{\partial^2 W}{\partial\dfrac{\partial\psi}{\partial r}\,\partial\dfrac{\partial\psi}{\partial z}}\right\}^2=\frac{1}{r^2};$$

and, consequently, Brunacci's criteria are satisfied. Thus in this case the theorem takes a form exactly similar in all respects to the form taken by it in the previous case.

6. We now pass on to the general case, and it will be necessary to recapitulate briefly some of Clebsch's results. He begins by taking two quantities a and b, which are functions of x, y, z and in general of the time also. He then assumes

$$u=\frac{\partial(a,b)}{\partial(y,z)}=\Delta_1,\quad v=\frac{\partial(a,b)}{\partial(z,x)}=\Delta_2,\quad w=\frac{\partial(a,b)}{\partial(x,y)}=\Delta_3.$$

This is admissible since these values give

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0,$$

and thus the equation of continuity is satisfied. He then shews that in the case of steady motion, *i.e.* when a and b are independent of the time, the energy integral takes the form

$$\frac{p}{\rho}+V+\tfrac{1}{2}q^2+F(a,b)=0,$$

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where $F(a, b)$ denotes an arbitrary function of a and b. He also shews that the hydrodynamical equations give rise to the two equations

$$\frac{\partial}{\partial x}\left\{\Delta_3 \frac{\partial b}{\partial y} - \Delta_2 \frac{\partial b}{\partial z}\right\} + \frac{\partial}{\partial y}\left\{\Delta_1 \frac{\partial b}{\partial z} - \Delta_3 \frac{\partial b}{\partial x}\right\} + \frac{\partial}{\partial z}\left\{\Delta_2 \frac{\partial b}{\partial x} - \Delta_1 \frac{\partial b}{\partial y}\right\} + \frac{\partial F}{\partial a} = 0,$$

and

$$\frac{\partial}{\partial x}\left\{\Delta_3 \frac{\partial a}{\partial z} - \Delta_2 \frac{\partial a}{\partial y}\right\} + \frac{\partial}{\partial y}\left\{\Delta_2 \frac{\partial a}{\partial x} - \Delta_1 \frac{\partial a}{\partial z}\right\} + \frac{\partial}{\partial z}\left\{\Delta_1 \frac{\partial a}{\partial y} - \Delta_2 \frac{\partial a}{\partial x}\right\} + \frac{\partial F}{\partial b} = 0.$$

Consider now the expression

$$U = \iiint \{\tfrac{1}{2}(\Delta_1^2 + \Delta_2^2 + \Delta_3^2) - F(a, b)\}\, dx\, dy\, dz.$$

Taking the variation of this expression we have

$$\begin{aligned}
\delta U = \iiint \Big[& \Delta_1 \left\{\frac{\partial a}{\partial y}\frac{\partial \delta b}{\partial z} + \frac{\partial b}{\partial z}\frac{\partial \delta a}{\partial y} - \frac{\partial a}{\partial z}\frac{\partial \delta b}{\partial y} - \frac{\partial b}{\partial y}\frac{\partial \delta a}{\partial z}\right\} \\
& + \Delta_2 \left\{\frac{\partial a}{\partial z}\frac{\partial \delta b}{\partial x} + \frac{\partial b}{\partial x}\frac{\partial \delta a}{\partial z} - \frac{\partial a}{\partial x}\frac{\partial \delta b}{\partial z} - \frac{\partial b}{\partial z}\frac{\partial \delta a}{\partial x}\right\} \\
& + \Delta_3 \left\{\frac{\partial a}{\partial x}\frac{\partial \delta b}{\partial y} + \frac{\partial b}{\partial y}\frac{\partial \delta a}{\partial x} - \frac{\partial a}{\partial y}\frac{\partial \delta b}{\partial x} - \frac{\partial b}{\partial x}\frac{\partial \delta a}{\partial y}\right\} \\
& - \left\{\frac{\partial F}{\partial a}\delta a + \frac{\partial F}{\partial b}\delta b\right\}\Big]\, dx\, dy\, dz \\
= \iint \delta a \Big\{ & l\left(\Delta_3 \frac{\partial b}{\partial y} - \Delta_2 \frac{\partial b}{\partial z}\right) \\
& + m\left(\Delta_1 \frac{\partial b}{\partial z} - \Delta_3 \frac{\partial b}{\partial x}\right) + n\left(\Delta_2 \frac{\partial b}{\partial x} - \Delta_1 \frac{\partial b}{\partial y}\right)\Big\}\, dS \\
+ \iint \delta b \Big\{ & l\left(\Delta_2 \frac{\partial a}{\partial z} - \Delta_3 \frac{\partial a}{\partial y}\right) \\
& + m\left(\Delta_3 \frac{\partial a}{\partial x} - \Delta_1 \frac{\partial a}{\partial z}\right) + n\left(\Delta_1 \frac{\partial a}{\partial y} - \Delta_2 \frac{\partial a}{\partial x}\right)\Big\}\, dS
\end{aligned}$$

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$$-\iiint \delta a \left[\frac{\partial}{\partial x} \left\{ \Delta_3 \frac{\partial b}{\partial y} - \Delta_2 \frac{\partial b}{\partial z} \right\} + \frac{\partial}{\partial y} \left\{ \Delta_1 \frac{\partial b}{\partial z} - \Delta_3 \frac{\partial b}{\partial x} \right\} + \frac{\partial}{\partial z} \left\{ \Delta_2 \frac{\partial b}{\partial x} - \Delta_1 \frac{\partial b}{\partial y} \right\} + \frac{\partial F}{\partial a} \right] dx\,dy\,dz$$

$$-\iiint \delta b \left[\left\{ \frac{\partial}{\partial z} \Delta_3 \frac{\partial a}{\partial z} - \Delta_3 \frac{\partial a}{\partial y} \right\} + \frac{\partial}{\partial y} \left\{ \Delta_3 \frac{\partial a}{\partial x} - \Delta_1 \frac{\partial a}{\partial z} \right\} + \frac{\partial}{\partial z} \left\{ \Delta_1 \frac{\partial a}{\partial y} - \Delta_2 \frac{\partial a}{\partial x} \right\} + \frac{\partial F}{\partial b} \right] dx\,dy\,dz.$$

This follows by Green's Theorem, the surface integrals being taken over the bounding surfaces of the space throughout which the volume integral is taken, and l, m, n being the direction cosines of the normal to the bounding surface drawn outwards from the fluid.

It will be seen that the equations given above cause the triple integrals to vanish, and that the double integrals will vanish also if the values of a and b be fixed over the boundaries.

It is also to be remarked that if the region throughout which U is taken be multiply connected, then it will also be necessary that the cyclic constants of a and b corresponding to the various circuits must be given also.

7. If the values of a and b be known over the boundary, then it is possible to calculate the normal velocity at the boundary. We will bring this fact into evidence by developing a property analogous to one possessed by the case of plane steady motion, viz. that the rate of variation of the stream function along the boundary measures the normal velocity at the boundary.

By means of the equation of the bounding surface we may eliminate z from the expressions for a and b as given over that surface, and thus express these quantities as functions of x and y only. We will distinguish the differential coefficients, with respect to x and y, of the said quantities, so expressed, by enclosing them in square brackets. Now the area of an infinitesimal portion of the bounding surface may be expressed in either of the forms $h\,da\,db$ or $k\,dx\,dy$, and these two forms are connected by the equation

$$da\,db = \left\{ \left[\frac{\partial a}{\partial x} \right] \left[\frac{\partial b}{\partial y} \right] - \left[\frac{\partial a}{\partial y} \right] \left[\frac{\partial b}{\partial x} \right] \right\} dx\,dy.$$

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And we have

$$\left[\frac{\partial a}{\partial x}\right]\left[\frac{\partial b}{\partial y}\right]-\left[\frac{\partial a}{\partial y}\right]\left[\frac{\partial b}{\partial x}\right]=\left\{\frac{\partial a}{\partial x}+\frac{\partial a}{\partial z}\frac{\partial z}{\partial x}\right\}\left\{\frac{\partial b}{\partial y}+\frac{\partial b}{\partial z}\frac{\partial z}{\partial y}\right\}$$
$$-\left\{\frac{\partial a}{\partial y}+\frac{\partial a}{\partial z}\frac{\partial z}{\partial y}\right\}\left\{\frac{\partial b}{\partial x}+\frac{\partial b}{\partial z}\frac{\partial z}{\partial x}\right\}$$
$$=\frac{\partial(a,b)}{\partial(x,y)}-\frac{\partial(a,b)}{\partial(y,z)}\frac{\partial z}{\partial x}-\frac{\partial(a,b)}{\partial(z,x)}\frac{\partial z}{\partial y}.$$

Further if $\phi(x, y, z)=0$ be the equation of the bounding surface, we have

$$\frac{\partial z}{\partial x}=-\frac{\partial\phi}{\partial x}\Big/\frac{\partial\phi}{\partial z}\text{ and }\frac{\partial z}{\partial y}=-\frac{\partial\phi}{\partial y}\Big/\frac{\partial\phi}{\partial z}.$$

Thus we obtain

$$da\,db\,\frac{\partial\phi}{\partial z}=\left\{\frac{\partial\phi}{\partial x}\frac{\partial(a,b)}{\partial(y,z)}+\frac{\partial\phi}{\partial y}\frac{\partial(a,b)}{\partial(z,x)}+\frac{\partial\phi}{\partial z}\frac{\partial(a,b)}{\partial(x,y)}\right\}dx\,dy.$$

And, denoting the direction cosines of the normal to the bounding surface by l, m, n, we have, since $dx\,dy=n\,dS$,

$$\frac{da\,db}{dS}=lu+mv+nw.$$

Conversely, if the value of the normal velocity be given over the boundary, we shall have a differential equation connecting the values of a and b at the boundary. This, however, will not be sufficient to determine the said values. But it must be remembered that the quantities a and b have not been fully defined as the stream functions in the two former cases were. The only restriction that has been put upon these quantities is that the surfaces represented by the equations

$$a=\text{const. and }b=\text{const.}$$

possess the property that if we draw a stream line through any point of one of them it will lie wholly on the surface. In fact it is clear that we may find an indefinite number of pairs of values for a and b that will express a given state of motion. Thus it would appear that the other condition to be fulfilled by the values of a and b at the bounding surface is of an arbitrary character; and thus it would seem that the condition that the normal velocity at the boundary is given is sufficient to render the motion determinate. The only

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other quantity connected with the motion, which is related to the normal at the boundary, is the spin about that line at the point of the boundary through which it is drawn. The value of this quantity does not seem to be calculable from a knowledge of a and b over the boundary.

8. It will be found that we cannot in the general case, as in the two special cases we have discussed, pick out a series of cases in which a solution must necessarily give rise to a minimum value of the integral in question. Clebsch has dealt with the discrimination of cases for multiple integrals involving more than one dependent variable; but as we have no work of the nature of Mr. Culverwell's in connection with these cases it will not repay to pursue the subject further.

ON SOME DIFFERENTIAL EQUATIONS SATISFIED BY THE QUANTITIES K, E, &c., IN ELLIPTIC FUNCTIONS.

By *Joseph Kleiber.*

§ 1. It is known* that the complete elliptic integrals K, K' satisfy the linear differential equation of the second order

$$(1-k^2)\frac{d^2u}{dk^2}+\frac{1-3k^2}{k}\frac{du}{dk}-u=0,$$

viz., that the general solution of this equation is

$$u=c_1K+c_2K'.$$

This equation is, as has been shewn by Dr. J. W. L. Glaisher†, one of a system of similar equations satisfied by the quantities K, K', E, E', I, I', G, G', where

$$I=E-K, \quad G=E-k'^2K,$$

the I being $-J$ of Weierstrass and G a new fundamental quantity, and the accented letters denoting always the same functions of k' as the unaccented are of k.

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* Cayley, *Elliptic Functions*, p. 50.

† *Quarterly Journal*, vol. xx., No. 10. Also *Proc. Camb. Phil. Soc.*, vol. v., Part iii.

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These differential equations, being transformed to the independent variable $h = k^2$, assume the form

$$4hh' \frac{d^2u}{dh^2} + 4(h' - h)\frac{du}{dh} + u = 0,$$

$$u = c_1 K + c_2 K';$$

$$4hh' \frac{d^2u}{dh^2} + 4h' \frac{du}{dh} + u = 0,$$

$$u = c_1 E + c_2 I';$$

$$4hh' \frac{d^2u}{dh^2} - 4h \frac{du}{dh} + u = 0,$$

$$u = c_1 I + c_2 E';$$

$$4hh' \frac{d^2u}{dh^2} - u = 0,$$

$$u = c_1 G + c_2 G';$$

where the inter-dependence of E and I and the independence of K and G should be noticed.

The differential equation satisfied by K, E, &c., the modular angle being the independent variable.

§ 2. I have found that these equations can be put in a remarkably symmetrical form, if we take the modular angle θ ($k = \sin\theta$) as the independent variable.

We have for this transformation the formulæ

$$\frac{du}{dh} = \frac{du}{d\theta} \operatorname{cec} 2\theta,$$

$$\frac{d^2u}{dh^2} = \frac{d^2u}{d\theta^2} \operatorname{cec}^2 2\theta - \frac{du}{dh} 2 \cot 2\theta \operatorname{cec}^2 2\theta.$$

Substituting these values into the differential equations, §1, we find the following system, where we draw the attention to the exact correspondence of the equation for G to that satisfied by K:

$$\frac{d^2u}{d\theta^2} + 2\cot 2\theta \frac{du}{d\theta} - u = 0 \ldots\ldots\ldots\ldots\ldots (1),$$

$$u = c_1 K + c_2 K';$$

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$$\frac{d^2u}{d\theta^2} - 2\cot 2\theta \frac{du}{d\theta} - u = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (2),$$

$$u = c_1 G + c_2 G';$$

$$\frac{d^2u}{d\theta^2} + 2\operatorname{cec} 2\theta \frac{du}{d\theta} + u = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (3),$$

$$u = c_1 E + c_2 I';$$

$$\frac{d^2u}{d\theta^2} - 2\operatorname{cec} 2\theta \frac{du}{d\theta} + u = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (4),$$

$$u = c_1 I + c_2 E'.$$

Differential relations connecting K, E, &c.

§ 3. If we transform to the new variables the relations

$$\frac{dK}{dh} = \frac{G}{2hh'}, \qquad \frac{dG}{dh} = \frac{K}{2},$$

$$\frac{dE}{dh} = \frac{I}{2h}, \qquad \frac{dI}{dh} = -\frac{E}{2h'};$$

and the corresponding relations holding good for K', G', E', I', we obtain the following symmetrical system:

$$\frac{dK}{d\theta} = 2G \operatorname{cec} 2\theta \quad \ldots\ldots\ldots\ldots\ldots\ldots (5),$$

$$\frac{d2G}{d\theta} = K \sin 2\theta \quad \ldots\ldots\ldots\ldots\ldots\ldots (6),$$

$$\frac{dE}{d\theta} = I \cot \theta \quad \ldots\ldots\ldots\ldots\ldots\ldots (7),$$

$$\frac{dI}{d\theta} = -E \tan \theta \quad \ldots\ldots\ldots\ldots\ldots\ldots (8);$$

and for the accented letters

$$\frac{dK'}{d\theta} = -2G' \operatorname{cec} 2\theta \quad \ldots\ldots\ldots\ldots\ldots\ldots (9),$$

$$\frac{d2G'}{d\theta} = -K' \sin 2\theta \quad \ldots\ldots\ldots\ldots\ldots\ldots (10),$$

$$\frac{dE'}{d\theta} = -I' \tan \theta \quad \ldots\ldots\ldots\ldots\ldots\ldots (11),$$

$$\frac{dI'}{d\theta} = E' \cot \theta \quad \ldots\ldots\ldots\ldots\ldots\ldots (12).$$

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It may be noticed that all these relations can be put into a form involving only $\tan\theta$, thus

$$\frac{dK}{d\theta} = G \div (\tan\theta + 1 \div \tan\theta),$$

$$\frac{dG}{d\theta} = K \quad (\tan\theta + 1 \div \tan\theta),$$

$$\frac{dE}{d\theta} = I \div \tan\theta,$$

$$\frac{dI}{d\theta} = -E \div \tan\theta;$$

and similarly for K', G', E', I'.

Multiplying the corresponding formulæ into one another we obtain the following remarkable relations:

$$\frac{dK}{d\theta}\frac{dG}{d\theta} - KG = 0,$$

$$\frac{dE}{d\theta}\frac{dI}{d\theta} + EI = 0,$$

$$\frac{dK'}{d\theta}\frac{dG'}{d\theta} - K'G' = 0,$$

$$\frac{dE'}{d\theta}\frac{dI'}{d\theta} + E'I' = 0;$$

which may be best written in the following way:

$$\frac{dK}{Kd\theta}\frac{dG}{Gd\theta} = 1 \ldots\ldots\ldots\ldots\ldots\ldots(13),$$

$$\frac{dE}{Ed\theta}\frac{dI}{Id\theta} = -1 \ldots\ldots\ldots\ldots\ldots\ldots(14),$$

$$\frac{dK'}{K'd\theta}\frac{dG'}{G'd\theta} = 1 \ldots\ldots\ldots\ldots\ldots\ldots(15),$$

$$\frac{dE'}{E'd\theta}\frac{dI'}{I'd\theta} = -1 \ldots\ldots\ldots\ldots\ldots\ldots(16);$$

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and these may be combined to give the following two "harmonic" formulæ:

$$\frac{dK}{Kd\theta}\frac{dG}{Gd\theta}\frac{dE}{Ed\theta}\frac{dI}{Id\theta} = -1 \quad \ldots\ldots\ldots\ldots(17),$$

$$\frac{dK'}{K'd\theta}\frac{dG'}{G'd\theta}\frac{dE'}{E'd\theta}\frac{dI'}{I'd\theta} = -1 \quad \ldots\ldots\ldots\ldots(18).$$

The relations 13–16 contain two quantities of the set K, G, E, I, each involved in the form of logarithmic differentials. In the following set all the four quantities appear together with the independent variable:

$$\frac{dK}{d\theta}\frac{dE}{d\theta} = GI\operatorname{cec}^2\theta \quad \ldots\ldots\ldots\ldots(19),$$

$$\frac{dG}{d\theta}\frac{dI}{d\theta} = -KE\sin^2\theta \quad \ldots\ldots\ldots\ldots(20),$$

$$\frac{dE}{d\theta}\frac{dG}{d\theta} = IK\cos^2\theta \quad \ldots\ldots\ldots\ldots(21),$$

$$\frac{dI}{d\theta}\frac{dK}{d\theta} = -EG\sec^2\theta \quad \ldots\ldots\ldots\ldots(22);$$

with the corresponding set

$$\frac{dK'}{d\theta}\frac{dE'}{d\theta} = G'I'\cos^2\theta \quad \ldots\ldots\ldots\ldots(23),$$

$$\frac{dG'}{d\theta}\frac{dI'}{d\theta} = -K'E'\sec^2\theta \quad \ldots\ldots\ldots\ldots(24),$$

$$\frac{dE'}{d\theta}\frac{dG'}{d\theta} = I'K'\operatorname{cec}^2\theta \quad \ldots\ldots\ldots\ldots(25),$$

$$\frac{dI'}{d\theta}\frac{dK'}{d\theta} = -E'G'\sin^2\theta \quad \ldots\ldots\ldots\ldots(26).$$

Again, taking other combinations of the same formulæ 5–12, we have

$$\frac{dK}{d\theta}\frac{dK'}{d\theta} = -GG' \quad \ldots\ldots\ldots\ldots(27),$$

$$\frac{dG}{d\theta}\frac{dG'}{d\theta} = -KK' \quad \ldots\ldots\ldots\ldots(28),$$

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$$\frac{dE}{d\theta}\frac{dE'}{d\theta} = -\,II' \quad \ldots\ldots\ldots\ldots\ldots\ldots (29),$$

$$\frac{dI}{d\theta}\frac{dI'}{d\theta} = -\,EE' \quad \ldots\ldots\ldots\ldots\ldots\ldots (30);$$

and also

$$\frac{dK}{d\theta}\frac{dG'}{d\theta} = -\,K'G \quad \ldots\ldots\ldots\ldots\ldots\ldots (31),$$

$$\frac{dK'}{d\theta}\frac{dG}{d\theta} = -\,KG' \quad \ldots\ldots\ldots\ldots\ldots\ldots (32),$$

$$\frac{dE}{d\theta}\frac{dI'}{d\theta} = +\,E'I \quad \ldots\ldots\ldots\ldots\ldots\ldots (33),$$

$$\frac{dE'}{d\theta}\frac{dI}{d\theta} = +\,EI' \quad \ldots\ldots\ldots\ldots\ldots\ldots (34).$$

Now, while the relations (27), (28), (33), (34) must remain as they stand, the series 29–32 may be put in another form, in which they involve only two of the functions instead of four.

For, from the well-known formula

$$KE' + K'E - KK' = \tfrac{1}{2}\pi,$$

we have also

$$EE' - II' = \tfrac{1}{2}\pi,$$

$$KG' + K'G = \tfrac{1}{2}\pi;$$

and by means of these the formulæ 29–32 assume the new form:

$$\frac{dE}{d\theta}\frac{dE'}{d\theta} + EE' = \tfrac{1}{2}\pi \quad \ldots\ldots\ldots\ldots\ldots\ldots (35),$$

$$\frac{dI}{d\theta}\frac{dI'}{d\theta} + II' = -\tfrac{1}{2}\pi \quad \ldots\ldots\ldots\ldots\ldots\ldots (36),$$

$$\frac{dK}{d\theta}\frac{dG'}{d\theta} - KG' = \tfrac{1}{2}\pi \quad \ldots\ldots\ldots\ldots\ldots\ldots (37),$$

$$\frac{dK'}{d\theta}\frac{dG}{d\theta} - K'G = \tfrac{1}{2}\pi \quad \ldots\ldots\ldots\ldots\ldots\ldots (38).$$

Thus, while in the majority of cases the quantities E and I are inter-dependent, while K and G appear as self-sufficient, the converse is the case with these relations, in which E and I appear as independent of one another, while K and G are involved together.

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The quantities W and H.

§ 5. In the papers quoted, § 1, Dr. Glaisher introduces three new symbols U, V, W, defined by the formulæ

$$U = \tfrac{1}{2}(I + G),$$
$$V = \tfrac{1}{2}(G + E),$$
$$W = \tfrac{1}{2}(I + E);$$

U, V being in some respects analogous to E and I, while W is analogous to K. But I do not find for U and V simple differential equations, when expressed in θ, and they do not satisfy the fundamental relation given below, § 7, which is satisfied by all other quantities hitherto considered. Again, W being analogous to K, leads to another quantity, which I denote by H, and which is connected with W by differential relations of the same form as those connecting K and G, and, as it will be seen further on, this quantity H satisfies differential equations corresponding to those satisfied by W in the same way as the equation satisfied by K and G correspond to each other.

The quantity G is connected with E and K by the formula

$$G = E - k'^2 K,$$

or

$$G = E - \cos^2\theta . K.$$

In the same way we have

$$H = E - \cos^2\theta . 2W,$$

or

$$= E - E\cos^2\theta - I\cos^2\theta,$$

$$H = E\sin^2\theta - I\cos^2\theta \quad \ldots\ldots\ldots\ldots\ldots\ldots (39),$$

in which form it appears to be very much like G, which is

$$G = E\sin^2\theta + I\cos^2\theta.$$

Differential equations satisfied by the quantities W, H, and relations connecting them.

§ 6. Differentiating H and W with respect to θ, we find

$$\frac{dW}{d\theta} = -H\operatorname{cec} 2\theta \quad \ldots\ldots\ldots\ldots\ldots\ldots (40),$$

$$\frac{dH}{d\theta} = 3W\sin 2\theta \quad \ldots\ldots\ldots\ldots\ldots\ldots (41);$$

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where $F(a, b)$ denotes an arbitrary function of a and b. He also shews that the hydrodynamical equations give rise to the two equations

$$\frac{\partial}{\partial x}\left\{\Delta_3 \frac{\partial b}{\partial y} - \Delta_2 \frac{\partial b}{\partial z}\right\} + \frac{\partial}{\partial y}\left\{\Delta_1 \frac{\partial b}{\partial z} - \Delta_3 \frac{\partial b}{\partial x}\right\} + \frac{\partial}{\partial z}\left\{\Delta_2 \frac{\partial b}{\partial x} - \Delta_1 \frac{\partial b}{\partial y}\right\} + \frac{\partial F}{\partial a} = 0,$$

and

$$\frac{\partial}{\partial x}\left\{\Delta_2 \frac{\partial a}{\partial z} - \Delta_3 \frac{\partial a}{\partial y}\right\} + \frac{\partial}{\partial y}\left\{\Delta_3 \frac{\partial a}{\partial x} - \Delta_1 \frac{\partial a}{\partial z}\right\} + \frac{\partial}{\partial z}\left\{\Delta_1 \frac{\partial a}{\partial y} - \Delta_2 \frac{\partial a}{\partial x}\right\} + \frac{\partial F}{\partial b} = 0.$$

Consider now the expression

$$U = \iiint \{\tfrac{1}{2}(\Delta_1^2 + \Delta_2^2 + \Delta_3^2) - F(a, b)\}\, dx\, dy\, dz.$$

Taking the variation of this expression we have

$$\begin{aligned}
\delta U = \iiint \Big[& \Delta_1 \left\{\frac{\partial a}{\partial y}\frac{\partial \delta b}{\partial z} + \frac{\partial b}{\partial z}\frac{\partial \delta a}{\partial y} - \frac{\partial a}{\partial z}\frac{\partial \delta b}{\partial y} - \frac{\partial b}{\partial y}\frac{\partial \delta a}{\partial z}\right\} \\
& + \Delta_2 \left\{\frac{\partial a}{\partial z}\frac{\partial \delta b}{\partial x} + \frac{\partial b}{\partial x}\frac{\partial \delta a}{\partial z} - \frac{\partial a}{\partial x}\frac{\partial \delta b}{\partial z} - \frac{\partial b}{\partial z}\frac{\partial \delta a}{\partial x}\right\} \\
& + \Delta_3 \left\{\frac{\partial a}{\partial x}\frac{\partial \delta b}{\partial y} + \frac{\partial b}{\partial y}\frac{\partial \delta a}{\partial x} - \frac{\partial a}{\partial y}\frac{\partial \delta b}{\partial x} - \frac{\partial b}{\partial x}\frac{\partial \delta a}{\partial y}\right\} \\
& - \left\{\frac{\partial F}{\partial a}\delta a + \frac{\partial F}{\partial b}\delta b\right\}\Big]\, dx\, dy\, dz
\end{aligned}$$

$$\begin{aligned}
= \iint \delta a \Big\{ & l\left(\Delta_3 \frac{\partial b}{\partial y} - \Delta_2 \frac{\partial b}{\partial z}\right) \\
& + m\left(\Delta_1 \frac{\partial b}{\partial z} - \Delta_3 \frac{\partial b}{\partial x}\right) + n\left(\Delta_2 \frac{\partial b}{\partial x} - \Delta_1 \frac{\partial b}{\partial y}\right)\Big\}\, dS \\
+ \iint \delta b \Big\{ & l\left(\Delta_2 \frac{\partial a}{\partial z} - \Delta_3 \frac{\partial a}{\partial y}\right) \\
& + m\left(\Delta_3 \frac{\partial a}{\partial x} - \Delta_1 \frac{\partial a}{\partial z}\right) + n\left(\Delta_1 \frac{\partial a}{\partial y} - \Delta_2 \frac{\partial a}{\partial x}\right)\Big\}\, dS
\end{aligned}$$

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$$-\iiint \delta a \left[\frac{\partial}{\partial x} \left\{ \Delta_2 \frac{\partial b}{\partial y} - \Delta_3 \frac{\partial b}{\partial z} \right\} + \frac{\partial}{\partial y} \left\{ \Delta_1 \frac{\partial b}{\partial z} - \Delta_2 \frac{\partial b}{\partial x} \right\} + \frac{\partial}{\partial z} \left\{ \Delta_3 \frac{\partial b}{\partial x} - \Delta_1 \frac{\partial b}{\partial y} \right\} + \frac{\partial F}{\partial a} \right] dx\,dy\,dz$$

$$-\iiint \delta b \left[\left\{ \frac{\partial}{\partial z} \Delta_2 \frac{\partial a}{\partial z} - \Delta_3 \frac{\partial a}{\partial y} \right\} + \frac{\partial}{\partial y} \left\{ \Delta_3 \frac{\partial a}{\partial x} - \Delta_1 \frac{\partial a}{\partial z} \right\} + \frac{\partial}{\partial z} \left\{ \Delta_1 \frac{\partial a}{\partial y} - \Delta_2 \frac{\partial a}{\partial x} \right\} + \frac{\partial F}{\partial b} \right] dx\,dy\,dz.$$

This follows by Green's Theorem, the surface integrals being taken over the bounding surfaces of the space throughout which the volume integral is taken, and l, m, n being the direction cosines of the normal to the bounding surface drawn outwards from the fluid.

It will be seen that the equations given above cause the triple integrals to vanish, and that the double integrals will vanish also if the values of a and b be fixed over the boundaries.

It is also to be remarked that if the region throughout which U is taken be multiply connected, then it will also be necessary that the cyclic constants of a and b corresponding to the various circuits must be given also.

7. If the values of a and b be known over the boundary, then it is possible to calculate the normal velocity at the boundary. We will bring this fact into evidence by developing a property analogous to one possessed by the case of plane steady motion, viz. that the rate of variation of the stream function along the boundary measures the normal velocity at the boundary.

By means of the equation of the bounding surface we may eliminate z from the expressions for a and b as given over that surface, and thus express these quantities as functions of x and y only. We will distinguish the differential coefficients, with respect to x and y, of the said quantities, so expressed, by enclosing them in square brackets. Now the area of an infinitesimal portion of the bounding surface may be expressed in either of the forms $h\,da\,db$ or $k\,dx\,dy$, and these two forms are connected by the equation

$$da\,db = \left\{ \left[\frac{\partial a}{\partial x} \right] \left[\frac{\partial b}{\partial y} \right] - \left[\frac{\partial a}{\partial y} \right] \left[\frac{\partial b}{\partial x} \right] \right\} dx\,dy.$$

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And we have

$$\left[\frac{\partial a}{\partial x}\right]\left[\frac{\partial b}{\partial y}\right]-\left[\frac{\partial a}{\partial y}\right]\left[\frac{\partial b}{\partial x}\right]=\left\{\frac{\partial a}{\partial x}+\frac{\partial a}{\partial z}\frac{\partial z}{\partial x}\right\}\left\{\frac{\partial b}{\partial y}+\frac{\partial b}{\partial z}\frac{\partial z}{\partial y}\right\}$$
$$-\left\{\frac{\partial a}{\partial y}+\frac{\partial a}{\partial z}\frac{\partial z}{\partial y}\right\}\left\{\frac{\partial b}{\partial x}+\frac{\partial b}{\partial z}\frac{\partial z}{\partial x}\right\}$$
$$=\frac{\partial(a,b)}{\partial(x,y)}-\frac{\partial(a,b)}{\partial(y,z)}\frac{\partial z}{\partial x}-\frac{\partial(a,b)}{\partial(z,x)}\frac{\partial z}{\partial y}.$$

Further if $\phi(x, y, z)=0$ be the equation of the bounding surface, we have

$$\frac{\partial z}{\partial x}=-\frac{\partial\phi}{\partial x}\Big/\frac{\partial\phi}{\partial z} \text{ and } \frac{\partial z}{\partial y}=-\frac{\partial\phi}{\partial y}\Big/\frac{\partial\phi}{\partial z}.$$

Thus we obtain

$$da\,db\,\frac{\partial\phi}{\partial z}=\left\{\frac{\partial\phi}{\partial x}\frac{\partial(a,b)}{\partial(y,z)}+\frac{\partial\phi}{\partial y}\frac{\partial(a,b)}{\partial(z,x)}+\frac{\partial\phi}{\partial z}\frac{\partial(a,b)}{\partial(x,y)}\right\}dx\,dy.$$

And, denoting the direction cosines of the normal to the bounding surface by l, m, n, we have, since $dx\,dy=n\,dS$,

$$\frac{da\,db}{dS}=lu+mv+nw.$$

Conversely, if the value of the normal velocity be given over the boundary, we shall have a differential equation connecting the values of a and b at the boundary. This, however, will not be sufficient to determine the said values. But it must be remembered that the quantities a and b have not been fully defined as the stream functions in the two former cases were. The only restriction that has been put upon these quantities is that the surfaces represented by the equations

$$a=\text{const. and } b=\text{const.}$$

possess the property that if we draw a stream line through any point of one of them it will lie wholly on the surface. In fact it is clear that we may find an indefinite number of pairs of values for a and b that will express a given state of motion. Thus it would appear that the other condition to be fulfilled by the values of a and b at the bounding surface is of an arbitrary character; and thus it would seem that the condition that the normal velocity at the boundary is given is sufficient to render the motion determinate. The only

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other quantity connected with the motion, which is related to the normal at the boundary, is the spin about that line at the point of the boundary through which it is drawn. The value of this quantity does not seem to be calculable from a knowledge of a and b over the boundary.

8. It will be found that we cannot in the general case, as in the two special cases we have discussed, pick out a series of cases in which a solution must necessarily give rise to a minimum value of the integral in question. Clebsch has dealt with the discrimination of cases for multiple integrals involving more than one dependent variable; but as we have no work of the nature of Mr. Culverwell's in connection with these cases it will not repay to pursue the subject further.

ON SOME DIFFERENTIAL EQUATIONS SATISFIED BY THE QUANTITIES K, E, &c., IN ELLIPTIC FUNCTIONS.

By *Joseph Kleiber.*

§ 1. It is known* that the complete elliptic integrals K, K' satisfy the linear differential equation of the second order

$$(1-k^2)\frac{d^2u}{dk^2}+\frac{1-3k^2}{k}\frac{du}{dk}-u=0,$$

viz., that the general solution of this equation is

$$u=c_1K+c_2K'.$$

This equation is, as has been shewn by Dr. J. W. L. Glaisher†, one of a system of similar equations satisfied by the quantities K, K', E, E', I, I', G, G', where

$$I=E-K, \quad G=E-k'^2K,$$

the I being $-J$ of Weierstrass and G a new fundamental quantity, and the accented letters denoting always the same functions of k' as the unaccented are of k.

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* Cayley, *Elliptic Functions*, p. 50.

† *Quarterly Journal*, vol. xx., No. 10. Also *Proc. Camb. Phil. Soc.*, vol. v., Part iii.

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These differential equations, being transformed to the independent variable $h=k^2$, assume the form

$$4hh'\frac{d^2u}{dh^2}+4(h'-h)\frac{du}{dh}+u=0,$$

$$u=c_1K+c_2K';$$

$$4hh'\frac{d^2u}{dh^2}+\quad 4h'\quad \frac{du}{dh}+u=0,$$

$$u=c_1E+c_2I';$$

$$4hh'\frac{d^2u}{dh^2}-\quad 4h\quad \frac{du}{dh}+u=0,$$

$$u=c_1I+c_2E';$$

$$4hh'\frac{d^2u}{dh^2}\qquad\qquad -u=0,$$

$$u=c_1G+c_2G';$$

where the inter-dependence of E and I and the independence of K and G should be noticed.

The differential equation satisfied by K, E, &c., the modular angle being the independent variable.

§ 2. I have found that these equations can be put in a remarkably symmetrical form, if we take the modular angle θ ($k=\sin\theta$) as the independent variable.

We have for this transformation the formulæ

$$\frac{du}{dh}=\frac{du}{d\theta}\operatorname{cec}2\theta,$$

$$\frac{d^2u}{dh^2}=\frac{d^2u}{d\theta^2}\operatorname{cec}^2 2\theta-\frac{du}{dh}2\cot 2\theta\operatorname{cec}^2 2\theta.$$

Substituting these values into the differential equations, §1, we find the following system, where we draw the attention to the exact correspondence of the equation for G to that satisfied by K:

$$\frac{d^2u}{d\theta^2}+2\cot 2\theta\frac{du}{d\theta}-u=0\ldots\ldots\ldots\ldots\ldots(1),$$

$$u=c_1K+c_2K';$$

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$$\frac{d^2u}{d\theta^2} - 2\cot 2\theta \frac{du}{d\theta} - u = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (2),$$

$$u = c_1 G + c_2 G';$$

$$\frac{d^2u}{d\theta^2} + 2\operatorname{cec} 2\theta \frac{du}{d\theta} + u = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (3),$$

$$u = c_1 E + c_2 I';$$

$$\frac{d^2u}{d\theta^2} - 2\operatorname{cec} 2\theta \frac{du}{d\theta} + u = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (4),$$

$$u = c_1 I + c_2 E'.$$

Differential relations connecting K, E, &c.

§ 3. If we transform to the new variables the relations

$$\frac{dK}{dh} = \frac{G}{2hh'}, \qquad \frac{dG}{dh} = \frac{K}{2},$$

$$\frac{dE}{dh} = \frac{I}{2h}, \qquad \frac{dI}{dh} = -\frac{E}{2h'};$$

and the corresponding relations holding good for K', G', E', I', we obtain the following symmetrical system:

$$\frac{dK}{d\theta} = 2G \operatorname{cec} 2\theta \quad \ldots\ldots\ldots\ldots\ldots\ldots (5),$$

$$\frac{d2G}{d\theta} = K \sin 2\theta \quad \ldots\ldots\ldots\ldots\ldots\ldots (6),$$

$$\frac{dE}{d\theta} = I \cot \theta \quad \ldots\ldots\ldots\ldots\ldots\ldots (7),$$

$$\frac{dI}{d\theta} = -E \tan \theta \quad \ldots\ldots\ldots\ldots\ldots\ldots (8);$$

and for the accented letters

$$\frac{dK'}{d\theta} = -2G' \operatorname{cec} 2\theta \quad \ldots\ldots\ldots\ldots\ldots\ldots (9),$$

$$\frac{d2G'}{d\theta} = -K' \sin 2\theta \quad \ldots\ldots\ldots\ldots\ldots\ldots (10),$$

$$\frac{dE'}{d\theta} = -I' \tan \theta \quad \ldots\ldots\ldots\ldots\ldots\ldots (11),$$

$$\frac{dI'}{d\theta} = E' \cot \theta \quad \ldots\ldots\ldots\ldots\ldots\ldots (12).$$

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It may be noticed that all these relations can be put into a form involving only $\tan\theta$, thus

$$\frac{dK}{d\theta} = G \div (\tan\theta + 1 \div \tan\theta),$$

$$\frac{dG}{d\theta} = K \quad (\tan\theta + 1 \div \tan\theta),$$

$$\frac{dE}{d\theta} = I \div \tan\theta,$$

$$\frac{dI}{d\theta} = -E \div \tan\theta;$$

and similarly for K', G', E', I'.

Multiplying the corresponding formulæ into one another we obtain the following remarkable relations:

$$\frac{dK}{d\theta}\frac{dG}{d\theta} - KG = 0,$$

$$\frac{dE}{d\theta}\frac{dI}{d\theta} + EI = 0,$$

$$\frac{dK'}{d\theta}\frac{dG'}{d\theta} - K'G' = 0,$$

$$\frac{dE'}{d\theta}\frac{dI'}{d\theta} + E'I' = 0;$$

which may be best written in the following way:

$$\frac{dK}{Kd\theta}\frac{dG}{Gd\theta} = 1 \quad \ldots\ldots\ldots\ldots\ldots\ldots (13),$$

$$\frac{dE}{Ed\theta}\frac{dI}{Id\theta} = -1 \quad \ldots\ldots\ldots\ldots\ldots\ldots (14),$$

$$\frac{dK'}{K'd\theta}\frac{dG'}{G'd\theta} = 1 \quad \ldots\ldots\ldots\ldots\ldots\ldots (15),$$

$$\frac{dE'}{E'd\theta}\frac{dI'}{I'd\theta} = -1 \quad \ldots\ldots\ldots\ldots\ldots\ldots (16);$$

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and these may be combined to give the following two "harmonic" formulæ:

$$\frac{dK}{Kd\theta}\frac{dG}{Gd\theta}\frac{dE}{Ed\theta}\frac{dI}{Id\theta} = -1 \quad \ldots\ldots\ldots\ldots(17),$$

$$\frac{dK'}{K'd\theta}\frac{dG'}{G'd\theta}\frac{dE'}{E'd\theta}\frac{dI'}{I'd\theta} = -1 \quad \ldots\ldots\ldots\ldots(18).$$

The relations 13–16 contain two quantities of the set K, G, E, I, each involved in the form of logarithmic differentials. In the following set all the four quantities appear together with the independent variable:

$$\frac{dK}{d\theta}\frac{dE}{d\theta} = GI\,\mathrm{cec}^2\theta \quad \ldots\ldots\ldots\ldots(19),$$

$$\frac{dG}{d\theta}\frac{dI}{d\theta} = -KE\sin^2\theta \quad \ldots\ldots\ldots\ldots(20),$$

$$\frac{dE}{d\theta}\frac{dG}{d\theta} = IK\cos^2\theta \quad \ldots\ldots\ldots\ldots(21),$$

$$\frac{dI}{d\theta}\frac{dK}{d\theta} = -EG\sec^2\theta \quad \ldots\ldots\ldots\ldots(22);$$

with the corresponding set

$$\frac{dK'}{d\theta}\frac{dE'}{d\theta} = G'I'\cos^2\theta \quad \ldots\ldots\ldots\ldots(23),$$

$$\frac{dG'}{d\theta}\frac{dI'}{d\theta} = -K'E'\sec^2\theta \quad \ldots\ldots\ldots\ldots(24),$$

$$\frac{dE'}{d\theta}\frac{dG'}{d\theta} = I'K'\,\mathrm{cec}^2\theta \quad \ldots\ldots\ldots\ldots(25),$$

$$\frac{dI'}{d\theta}\frac{dK'}{d\theta} = -E'G'\sin^2\theta \quad \ldots\ldots\ldots\ldots(26).$$

Again, taking other combinations of the same formulæ 5–12, we have

$$\frac{dK}{d\theta}\frac{dK'}{d\theta} = -GG' \quad \ldots\ldots\ldots\ldots(27),$$

$$\frac{dG}{d\theta}\frac{dG'}{d\theta} = -KK' \quad \ldots\ldots\ldots\ldots(28),$$

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$$\frac{dE}{d\theta}\frac{dE'}{d\theta} = -\,II' \quad \ldots\ldots\ldots\ldots\ldots\ldots (29),$$

$$\frac{dI}{d\theta}\frac{dI'}{d\theta} = -\,EE' \quad \ldots\ldots\ldots\ldots\ldots\ldots (30);$$

and also

$$\frac{dK}{d\theta}\frac{dG'}{d\theta} = -\,K'G \quad \ldots\ldots\ldots\ldots\ldots\ldots (31),$$

$$\frac{dK'}{d\theta}\frac{dG}{d\theta} = -\,KG' \quad \ldots\ldots\ldots\ldots\ldots\ldots (32),$$

$$\frac{dE}{d\theta}\frac{dI'}{d\theta} = +\,E'I \quad \ldots\ldots\ldots\ldots\ldots\ldots (33),$$

$$\frac{dE'}{d\theta}\frac{dI}{d\theta} = +\,EI' \quad \ldots\ldots\ldots\ldots\ldots\ldots (34).$$

Now, while the relations (27), (28), (33), (34) must remain as they stand, the series 29–32 may be put in another form, in which they involve only two of the functions instead of four.

For, from the well-known formula

$$KE' + K'E - KK' = \tfrac{1}{2}\pi,$$

we have also

$$EE' - II' = \tfrac{1}{2}\pi,$$

$$KG' + K'G = \tfrac{1}{2}\pi;$$

and by means of these the formulæ 29–32 assume the new form:

$$\frac{dE}{d\theta}\frac{dE'}{d\theta} + EE' = \tfrac{1}{2}\pi \quad \ldots\ldots\ldots\ldots\ldots\ldots (35),$$

$$\frac{dI}{d\theta}\frac{dI'}{d\theta} + II' = -\tfrac{1}{2}\pi \quad \ldots\ldots\ldots\ldots\ldots\ldots (36),$$

$$\frac{dK}{d\theta}\frac{dG'}{d\theta} - KG' = \tfrac{1}{2}\pi \quad \ldots\ldots\ldots\ldots\ldots\ldots (37),$$

$$\frac{dK'}{d\theta}\frac{dG}{d\theta} - K'G = \tfrac{1}{2}\pi \quad \ldots\ldots\ldots\ldots\ldots\ldots (38).$$

Thus, while in the majority of cases the quantities E and I are inter-dependent, while K and G appear as self-sufficient, the converse is the case with these relations, in which E and I appear as independent of one another, while K and G are involved together.

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The quantities W and H.

§ 5. In the papers quoted, § 1, Dr. Glaisher introduces three new symbols U, V, W, defined by the formulæ

$$U = \tfrac{1}{2}(I + G),$$
$$V = \tfrac{1}{2}(G + E),$$
$$W = \tfrac{1}{2}(I + E);$$

U, V being in some respects analogous to E and I, while W is analogous to K. But I do not find for U and V simple differential equations, when expressed in θ, and they do not satisfy the fundamental relation given below, § 7, which is satisfied by all other quantities hitherto considered. Again, W being analogous to K, leads to another quantity, which I denote by H, and which is connected with W by differential relations of the same form as those connecting K and G, and, as it will be seen further on, this quantity H satisfies differential equations corresponding to those satisfied by W in the same way as the equation satisfied by K and G correspond to each other.

The quantity G is connected with E and K by the formula

$$G = E - k'^2 K,$$

or

$$G = E - \cos^2\theta . K.$$

In the same way we have

$$H = E - \cos^2\theta . 2W,$$

or

$$= E - E\cos^2\theta - I\cos^2\theta,$$

$$H = E\sin^2\theta - I\cos^2\theta \quad \ldots\ldots\ldots\ldots\ldots\ldots (39),$$

in which form it appears to be very much like G, which is

$$G = E\sin^2\theta + I\cos^2\theta.$$

Differential equations satisfied by the quantities W, H, and relations connecting them.

§ 6. Differentiating H and W with respect to θ, we find

$$\frac{dW}{d\theta} = -H\operatorname{cec} 2\theta \quad \ldots\ldots\ldots\ldots\ldots\ldots (40),$$

$$\frac{dH}{d\theta} = 3W\sin 2\theta \quad \ldots\ldots\ldots\ldots\ldots\ldots (41);$$

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and, denoting by accented letters the same functions of k' as the unaccented are of k, we have also

$$\frac{dW'}{d\theta} = H' \operatorname{cec} 2\theta \quad \ldots\ldots\ldots\ldots\ldots\ldots (42),$$

$$\frac{dH'}{d\theta} = -3W' \sin 2\theta \quad \ldots\ldots\ldots\ldots\ldots\ldots (43);$$

and the differential equations of the second order satisfied by these quantities are

$$\frac{d^2u}{d\theta^2} + 2\cot 2\theta \frac{du}{d\theta} + 3u = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (44),$$

$$u = c_1 W + c_2 W';$$

$$\frac{d^2u}{d\theta^2} - 2\cot 2\theta \frac{du}{d\theta} + 3u = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (45),$$

$$u = c_1 H + c_2 H'.$$

While at the same time the following relations hold good:

$$\frac{dW}{Wd\theta}\frac{dH}{Hd\theta} = -3 \quad \ldots\ldots\ldots\ldots\ldots\ldots (46),$$

$$\frac{dW'}{W'd\theta}\frac{dH'}{H'd\theta} = -3 \quad \ldots\ldots\ldots\ldots\ldots\ldots (47);$$

$$\frac{dW}{d\theta}\frac{dW'}{d\theta} = -HH' \operatorname{cec}^2 2\theta \quad \ldots\ldots\ldots\ldots\ldots\ldots (48),$$

$$\frac{dH}{d\theta}\frac{dH'}{d\theta} = -9WW' \sin^2 2\theta \quad \ldots\ldots\ldots\ldots\ldots\ldots (49);$$

$$\frac{dW}{d\theta}\frac{dH'}{d\theta} = 3W'H \quad \ldots\ldots\ldots\ldots\ldots\ldots (50),$$

$$\frac{dW'}{d\theta}\frac{dH}{d\theta} = 3WH' \quad \ldots\ldots\ldots\ldots\ldots\ldots (51).$$

Corresponding to the formula

$$GK' + KG' = \tfrac{1}{2}\pi,$$

we have for the new quantities

$$2W.H' + 2W'.H = \tfrac{1}{2}\pi,$$

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by means of which the last two relations are transformed in the following, involving only two of the four letters W, W', H, H', each:

$$\frac{d.2W}{d\theta}\frac{dH'}{d\theta}+3.2W.H'=\tfrac{3}{2}\pi \quad \text{.....(53)},$$

$$\frac{d.2W'}{d\theta}\frac{dH}{d\theta}+3.2W'.H=\tfrac{3}{2}\pi \quad \text{.....(54)}.$$

(It will be noticed that we write $2W$ instead of W and sometimes $2G$ instead of G, these quantities giving more symmetrical results, and in fact W being defined as $\frac{1}{2}(E+I)$, $2W$ is simply $E+I$).

Differential equations of the first order satisfied by the ratios of correspondent pairs of the set K, E, &c., *viz. by*

$$\frac{K}{G}, \frac{E}{I}, \text{ \&c.}$$

§ 7. We have hitherto considered equations of the second order analagous to the known Legendre's equation satisfied by K and K', given in § 1, introducing merely a new independent variable, which reduced that equation and its analogues to a simpler form. We proceed now to shew that the *ratios* of correspondent pairs of the set K, E, I, G, W, H, &c. satisfy some very simple differential equations *of the first order*. By correspondent quantities, we mean those which are mutually connected by the differential relations (5—12) and (40—43).

All these relations are of the form

$$\frac{dy}{d\theta}=ax,$$

$$\frac{dx}{d\theta}=by,$$

where a, b are functions of θ only, *i.e.* they do not involve x or y. Now, if we put

$$u=\frac{x}{y},$$

we have by differentiation,

$$\frac{du}{d\theta}=\frac{1}{y}\frac{dx}{d\theta}-\frac{x}{y^2}\frac{dy}{d\theta}=b-au^2,$$

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i.e. the quantity satisfies the differential equation of the first order

$$\frac{du}{d\theta} + au^2 - b = 0.$$

And in the same way it may be shewn that the ratio $\frac{y}{x}$ satisfies the differential equation

$$\frac{du}{d\theta} + bu^2 - a = 0.$$

Hence, substituting for a, b their values for the six respective groups, we get the following set of equations:

$$\frac{du}{d\theta} + u^2 \sin 2\theta - \operatorname{cec} 2\theta = 0 \quad \ldots\ldots\ldots\ldots(55),$$

$$u = \frac{K}{2G};$$

$$\frac{du}{d\theta} + u^2 \operatorname{cec} 2\theta - \sin 2\theta = 0 \quad \ldots\ldots\ldots\ldots(56),$$

$$u = \frac{2G}{K};$$

$$\frac{du}{d\theta} + u^2 \cot\theta + \tan\theta = 0 \quad \ldots\ldots\ldots\ldots(57),$$

$$u = \frac{I}{E};$$

$$\frac{du}{d\theta} - u^2 \tan\theta - \cot\theta = 0 \quad \ldots\ldots\ldots\ldots(58),$$

$$u = \frac{E}{I};$$

$$\frac{du}{d\theta} - u^2 \operatorname{cec} 2\theta - 3 \sin 2\theta = 0 \quad \ldots\ldots\ldots\ldots(59),$$

$$u = \frac{H}{W};$$

$$\frac{du}{d\theta} + 3u^2 \sin 2\theta + \operatorname{cec} 2\theta = 0 \quad \ldots\ldots\ldots\ldots(60),$$

$$u = \frac{W}{H};$$

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and similarly for the accented letters

$$\frac{du}{d\theta} - u^2 \sin 2\theta + \operatorname{cec} 2\theta = 0 \quad \ldots\ldots\ldots\ldots (61),$$

$$u = \frac{K'}{2G'};$$

$$\frac{du}{d\theta} - u^2 \operatorname{cec} 2\theta + \sin 2\theta = 0 \quad \ldots\ldots\ldots\ldots (62),$$

$$u = \frac{2G'}{K'};$$

$$\frac{du}{d\theta} + u^2 \cot\theta + \tan H = 0 \quad \ldots\ldots\ldots\ldots (63),$$

$$u = \frac{E'}{I'};$$

$$\frac{du}{d\theta} - u^2 \tan\theta - \cot\theta = 0 \quad \ldots\ldots\ldots\ldots (64),$$

$$u = \frac{I'}{E'};$$

$$\frac{du}{d\theta} + u^2 \operatorname{cec} 2\theta + 3 \sin 2\theta = 0 \quad \ldots\ldots\ldots\ldots (65),$$

$$u = \frac{H'}{W'};$$

$$\frac{du}{d\theta} - 3u^2 \sin 2A - \operatorname{cec} 2\theta = 0 \quad \ldots\ldots\ldots\ldots (66),$$

$$u = \frac{W'}{H'}.$$

General solutions of the differential equations of the first order of § 7.

§ 8. Comparing the equations found in § 7 with each other, we see that *the same* equations are satisfied by $\frac{E}{I}$ and $\frac{I'}{E'}$, and similarly by $\frac{I}{E}$ and $\frac{E'}{I'}$, *i.e.* these values are particular integrals of the equations (57) and (58). At the same time, by merely changing the sign of u in the equations (61), (62),

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(65), (66), we reduce them to identity with (55), (56), (59), (60), so that $\frac{K}{2G}$ and $-\frac{K'}{2G'}$, $\frac{2G}{K}$ and $-\frac{2G'}{K'}$, $\frac{H}{W}$ and $-\frac{H'}{W'}$, $\frac{W}{H}$ and $-\frac{W'}{H'}$ are particular integrals of the four differential equations of the first order just mentioned.

It is of some interest to find *the general integrals* of those equations. But, as they are all of the second degree, it will be found convenient to transform them to linear differential equations of the second order, of which we well know, by what has been just said, two particular integrals, and will be thus able to construct the general solution. The transformation is performed on all by means of similiar substitutions. Taking as an example of it the equations satisfied by $\frac{I}{E}$ and $\frac{E'}{I'}$, we write

$$u = \frac{\tan\theta}{v}\frac{dv}{d\theta} \quad \ldots\ldots\ldots\ldots\ldots\ldots (67),$$

and by means of this substitution our equation is transformed into

$$\frac{d^2u}{d\theta^2} + 2\operatorname{cec} 2\theta \frac{dv}{d\theta} + v = 0.$$

Now this is an equation which we have already seen in § 2, viz. it is the equation (3) satisfied by

$$v = c_1 E + c_2 I'.$$

This being the general value of v, we find for u, if we put $\frac{c_2}{c_1} = c$,

$$u = \frac{I + cE'}{E + cI'} \quad \ldots\ldots\ldots\ldots\ldots\ldots (68),$$

and this is the required general solution of (57). For $c = 0$ it gives $\frac{I}{E}$, for $c = \infty$, $\frac{E'}{I'}$ as particular values, as it ought to give.

Another way of solving the equation in v leads to the same result. If the equation (3) were not given beforehand, we could proceed by ordinary methods in constructing the general solution of the differential equation of the second order in v by means of the two particular integrals of the equation in u. For, from (67) we have

$$v = e^{\int u \cot\theta \, d\theta},$$

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and knowing that v may be $\frac{I}{E}$ or $\frac{E'}{I'}$, this gives for the general value of v

$$v = c_1 e^{\int \frac{I}{E} \cot\theta \, d\theta} + c_2 e^{\int \frac{E'}{I'} \cot\theta \, d\theta}.$$

Now the differential relations (7) and (12) in § 12 give

$$I \cot\theta = \frac{dE}{d\theta}, \quad E' \cot\theta = \frac{dI'}{d\theta},$$

and substituting these quantities under the signs of integration they give $\frac{dE}{E}$, $\frac{dI'}{I'}$ respectively, so that, after integration, we have again

$$v = c_1 E + c_2 I',$$

and thus we are led to the same general solution (68).

Treating in the same way the equation (58), *i.e.* transforming it by the substitution

$$u = -\frac{\cot\theta}{v} \frac{dv}{d\theta}$$

into a liner differential equation of the second order, we find that this is our equation (4), which leads to the general solution

$$u = \frac{E + cI'}{I + cE'} \quad \ldots\ldots\ldots\ldots\ldots\ldots (69)$$

for (58).

And in the same way we find that

$$u = \frac{K - cK'}{2G + c2G'} \quad \ldots\ldots\ldots\ldots\ldots\ldots (70),$$

$$u = \frac{2G - c2G'}{K + cK'} \quad \ldots\ldots\ldots\ldots\ldots\ldots (71),$$

$$u = \frac{H - cH'}{W + cW'} \quad \ldots\ldots\ldots\ldots\ldots\ldots (72),$$

$$u = \frac{W - cW'}{H + cH'} \quad \ldots\ldots\ldots\ldots\ldots\ldots (73),$$

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as the general solutions of (55), (56), (59), (60). Thus our system of 12 equations (55)—(66) reduces to the following six:

$$\frac{du}{d\theta} + u^2 \sin 2\theta - \operatorname{cec} 2\theta = 0 \quad \ldots\ldots\ldots\ldots\ldots (74),$$

$$u = \frac{K - cK'}{2G + c2G'};$$

$$\frac{du}{d\theta} + u^2 \operatorname{cec} 2\theta - \sin 2\theta = 0 \quad \ldots\ldots\ldots\ldots\ldots (75),$$

$$u = \frac{2G - c2G'}{K + cK'};$$

$$\frac{du}{d\theta} + u^2 \cot\theta + \tan\theta = 0 \quad \ldots\ldots\ldots\ldots\ldots (76),$$

$$u = \frac{I + cE'}{E + cI'};$$

$$\frac{du}{d\theta} - u^2 \tan\theta - \operatorname{cec}\theta = 0 \quad \ldots\ldots\ldots\ldots\ldots (77),$$

$$u = \frac{E + cI'}{I + cE'};$$

$$\frac{du}{d\theta} - u^2 \operatorname{cec} 2\theta - 3 \sin 2\theta = 0 \quad \ldots\ldots\ldots\ldots\ldots (78),$$

$$u = \frac{H - cH'}{W + cW'};$$

$$\frac{du}{d\theta} + 3u^2 \sin 2\theta + \operatorname{cec} 2\theta = 0 \quad \ldots\ldots\ldots\ldots\ldots (79),$$

$$u = \frac{W - cW'}{H + cH'}.$$

The same differential equations of the first order with h or k as independent variable.

§ 9. The differential equations of the last section may be transformed to the following forms, in which h or k instead of θ are taken as the independent variable. Both systems present themselves in very simple forms:

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$$4h(1-h)\frac{du}{dh}+4h(1-h)u^2-1=0 \quad \ldots\ldots\ldots(80),$$

$$u=\frac{K-cK'}{2G+c2G'};$$

$$4h(1-h)\frac{du}{dh}+u^2-4h(1-h)=0 \quad \ldots\ldots\ldots(81),$$

$$u=\frac{2G-c2G'}{K+cK'};$$

$$2h(1-h)\frac{du}{dh}+(1-h)u^2+h=0 \quad \ldots\ldots\ldots(82),$$

$$u=\frac{I+cE'}{E+cI'};$$

$$2h(1-h)\frac{du}{dh}-hu'-(1-h)=0 \quad \ldots\ldots\ldots(83),$$

$$u=\frac{E+cI'}{I+cE'};$$

$$4h(1-h)\frac{du}{dh}-u^2-12h(1-h)=0 \quad \ldots\ldots\ldots(84),$$

$$u=\frac{H-cH'}{W+cW'};$$

$$4h(1-h)\frac{du}{dh}+12h(1-h)u^2+1=0 \quad \ldots\ldots\ldots(85),$$

$$u=\frac{W-cW'}{H+cH'}.$$

And if in order to avoid numerical coefficients we replace again G for $2G$ and take $2W$ instead of W, we have for k as the independent variable:

$$k(1-k^2)\frac{du}{dk}+k^2(1-k^2)^2u^2-1=0 \quad \ldots\ldots\ldots(86),$$

$$u=\frac{K-cK'}{G+cG'};$$

$$k(1-k^2)\frac{du}{dk}+u^2-k^2(1-k^2)=0 \quad \ldots\ldots\ldots(87),$$

$$u=\frac{G-cG'}{K+cK'};$$

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$$k(1-k^2)\frac{du}{dk}+(1-k^2)u^2+k^2=0 \quad \ldots\ldots\ldots(88),$$

$$u=\frac{I+cE'}{E+cI'};$$

$$k(1-k^2)\frac{du}{dk}-k^2u^2-(1-k^2)=0 \quad \ldots\ldots\ldots(89),$$

$$u=\frac{E+cI'}{I+cE'};$$

$$k(1-k^2)\frac{du}{dk}-u^2-3k^2(1-k^2)=0 \quad \ldots\ldots\ldots(90),$$

$$u=\frac{H+cH'}{2W+c2W'};$$

$$k(1-k^2)\frac{du}{dk}+3k^2(1-k^2)u^2+1=0 \quad \ldots\ldots\ldots(91),$$

$$u=\frac{2W-c2W'}{H+cH'}.$$

The differential equations of the second order for K, E, &c. in the normal form.

§ 10. It is known that any linear differential equation of the second order of the form

$$\frac{d^2y}{dx^2}+2P\frac{dy}{dx}+Qy=0$$

may be reduced by the substitution

$$y=ve^{-\int Pdx}$$

to the "normal form"

$$\frac{d^2v}{dx^2}+Iv=0$$

where I is the invariant

$$I=Q-\frac{dP}{dx}-P^2.$$

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If we apply this transformation to our equations § 2, we find the following system:

$$\frac{d^2u}{d\theta^2} + \operatorname{cec}^2\theta u = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (92),$$

$$u = c_1 K\sqrt{(\operatorname{cec} 2\theta)} + c_2 K'\sqrt{(\operatorname{cec} 2\theta)};$$

$$\frac{d^2u}{d\theta^2} - 3\operatorname{cec}^2\theta u = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (93),$$

$$u = c_1 G\sqrt{(\sin 2\theta)} + c_2 G'\sqrt{(\sin 2\theta)};$$

$$\frac{d^2u}{d\theta^2} + (\operatorname{cec}^2\theta - \tan^2\theta)\, u = 0 \quad \ldots\ldots\ldots\ldots (94),$$

$$u = c_1 E\sqrt{(\cot\theta)} + c_2 I'\sqrt{(\tan\theta)};$$

$$\frac{d^2u}{d\theta^2} + (3\operatorname{cec}^2 2\theta - \tan^2\theta)\, u = 0 \quad \ldots\ldots\ldots (95),$$

$$u = c_1 I\sqrt{(\tan\theta)} + c_2 E'\sqrt{(\cot\theta)};$$

$$\frac{d^2u}{d\theta^2} + (4 + \operatorname{cec}^2 2\theta)\, u = 0 \quad \ldots\ldots\ldots\ldots\ldots (96),$$

$$u = c_1 W\sqrt{(\operatorname{cec} 2\theta)} + c_2 W'\sqrt{(\operatorname{cec} 2\theta)};$$

$$\frac{d^2u}{d\theta^2} + (4 - 3\operatorname{cec}^2 2\theta)\, u = 0 \quad \ldots\ldots\ldots\ldots\ldots (97),$$

$$u = c_1 H\sqrt{(\sin 2\theta)} + c_2 H'\sqrt{(\sin 2\theta)}.$$

Remarks on the preceding formulæ, and hypothetical relations between two new quantities M and N, &c.

§ 11. The fundamental relations § 7 between the quantities K and G, E and I, H and W, are, in their general form independent of the variable with respect to which the differentials are taken; but while in any other system there is no particular connexion between a and b of the formulæ

$$\frac{dy}{d\theta} = ax, \quad \frac{dx}{d\theta} = by,$$

satisfied by the quantities mentioned, we have, if θ be the independent variable

$$ab = \text{constant},$$

with different values for different pairs of quantities,

$ab = 1$ for K and G,

$ab = -1$ for E and I,

$ab = -3$ for W and H.

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This suggests that there is some other pair of quantities, say M and N, analogous to E and I, and bearing to them the same relation as H and W bear to G and K, and for which

$$ab = +3.$$

But if we construct new quantities by means of relations analogous to those connecting E and I with K and G, *i.e.* if starting from the relations

$$E = G + \cos^2\theta K,$$
$$I = G - \sin^2\theta K,$$

we calculate the values of

$$H + \cos^2\theta\, 2W,$$
$$H - \sin^2\theta\, 2W,$$

we are thrown back on known quantities, the first of these expressions being equal to E, the second to $-I$.

The following "hypothetical" formulæ will hold good for those suspected linear combinations of E and K (or, what is the same thing of H and W),

$$\frac{d^2u}{d\theta^2} + 2\operatorname{cec} 2\theta \frac{du}{d\theta} - 3u = 0 \quad \ldots\ldots\ldots\ldots (98),$$
$$u = c_1 M + c_2 N';$$

$$\frac{d^2u}{d\theta^2} - 2\operatorname{cec} 2\theta \frac{du}{d\theta} - 3u = 0 \quad \ldots\ldots\ldots\ldots (99),$$
$$u = c_1 N + c_2 M';$$

$$\frac{du}{d\theta} + 3u^2 \cot\theta - \tan\theta = 0 \quad \ldots\ldots\ldots\ldots (100),$$
$$u = \frac{M + cN'}{N + cM'};$$

$$\frac{du}{d\theta} + u^2 \tan\theta - 3\cot\theta = 0 \quad \ldots\ldots\ldots\ldots (101),$$
$$u = \frac{N + cM'}{M + cN'};$$

and formulæ analogous to the systems (7), (8); (11), (12); (14), (16); (17), (18); (19)–(26); (29), (30); (33), (34); (35), (36), &c. will hold good for these quantities M and N.

I have obtained an interesting geometrical interpretation of some of the formulæ contained in this paper. This interpretation, however, I must defer to a future number.

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ON THE MOTION OF LIQUID IN A ROTATING SPHERICAL SECTOR.

By *G. C. Calliphronas, M.A.*

I PROPOSE to apply the method of sources to determine the motion of liquid filling a spherical sector, formed by two diametral planes of a sphere intersecting at an angle $\frac{\pi}{m}$ and the intercepted spherical surface, rotating about its edge.

Consider a spherical surface centre origin and radius a described in an infinite mass of liquid. Let the equations of $2m$ planes intersecting in the axis of z be

$$\phi = 0,\ \phi = \beta,\ \phi = 2\beta,\ \ldots,\ \phi = (2m-1)\beta,$$

and let us imagine liquid to be continuously generated over the planes

$$\phi - 0,\ \phi = 2\beta,\ \ldots,\ \phi = 2(m-1)\beta,$$

at a rate $2\omega c \sin\alpha$ per unit area per unit time (c, α being the distance from the origin and co-latitude of the source) and absorbed at a similar rate over the planes

$$\phi = \beta,\ \phi = 3\beta,\ \ldots,\ \phi = (2m-1)\beta;$$

c varying from 0 to a.

Then the normal velocity of the liquid over the semicircles $\phi = 0$, $\phi = \beta$ will be $\omega c \sin\alpha$, for it is easily seen that the normal velocity at either plane depends only on the sources or sinks in that plane, the symmetry of the sources and sinks in all the other planes neutralizing their contributions.

The question therefore is evidently reduced to the determination of the velocity potential of the motion produced in a spherical vessel by a source situated at one given point within it and a sink at another: since when we know this we can by integration deduce the velocity potential required.

Let then there be a unit source (that is one which discharges unit volume in unit time) at a point A within the sphere, centre O, and a unit sink at a point B. Let $OA = c$, $\cos POA = \mu$, P being any point within the sphere, let $Q_1, Q_2, \ldots$, be Laplace's coefficients with μ for variable and let accented quantities refer to B.

Consider the function V given by the equation

$$4\pi V = -\frac{1}{AP} - \Sigma \frac{n+1}{n} \frac{r^n c^n}{a^{2n+1}} Q_n + \frac{1}{BP} + \Sigma \frac{n+1}{n} \frac{r^n c'^n}{a^{2n+1}} Q_n',$$

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where $OP=r$ and a is the radius of the sphere, the summations extending to all positive integral values of n excluding zero.

We see that (1) when AP is indefinitely small the expression on the right reduces to $-\frac{1}{AP}$, (2) by expanding $\frac{1}{AP}$ in the series $\frac{1}{r}+Q_1\frac{c}{r^2}+\ldots$, and so for $\frac{1}{BP}$, the value of $\frac{dV}{dr}$ is zero when $r=a$.

Hence V is the velocity potential due to the given source and sink.

[The above expression can be put into the finite form

$$-\frac{1}{AP}-\frac{a}{(r^2c^2+a^4-2\mu rca^2)^{\frac{1}{2}}}+\frac{1}{a}\log\{a^2-\mu cr+(r^2c^2+a^4-2\mu rca^2)^{\frac{1}{2}}\}$$

+ corresponding expression for B, but the expansion is more convenient].

It is well to notice that the terms we add to $-\frac{1}{AP}$ to produce zero normal velocity when $r=a$ do not affect the motion in the immediate neighbourhood of the source.

Hence the required velocity potential for the rotating sector is obtained by adding the $2m$ expressions whose type is

$$-\frac{\omega}{2\pi}\int_0^\pi \sin\alpha\,d\alpha\int_0^a c^2dc\left\{\frac{1}{(r^2+c^2-2\mu rc)^{\frac{1}{2}}}+\Sigma\frac{n+1}{n}\frac{r^nc^n}{a^{2n+1}}Q_n\right\},$$

where $\mu=\cos\theta\cos\alpha+\sin\theta\sin\alpha\cos(\phi-k\beta)$,

the sign being changed for odd values of k.

Integrating with regard to c and bearing in mind that we must, in using the expanded form of $(r^2+c^2-2\mu rc)^{-\frac{1}{2}}$, divide the integration into two parts, namely

$$\int_0^r c^2dc\left(\frac{1}{r}+Q_1\frac{c}{r^2}+\ldots\right)+\int_r^a c^2dc\left(\frac{1}{c}+Q_1\frac{r}{c^2}+\ldots\right),$$

our typical expression becomes

$$-\frac{\omega}{2\pi}\int_0^\pi \sin\alpha\,d\alpha\left\{\frac{a^2}{2}-\frac{r^2}{6}+r^2Q_2\left(\tfrac{1}{2}+\log\frac{a}{r}\right)+r^2\Sigma\frac{2n+1}{(n-2)(n+3)}Q_n-2\Sigma\frac{2n+1}{n(n-2)(n+3)}\frac{r^n}{a^{n-2}}Q_n\right\},$$

the summations excluding $n=2$.

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Now Q_n expanded in tesseral harmonics takes the form

$$Q_n = P_n(\alpha)P_n(\theta) + \ldots + 2\frac{(n-\sigma)!}{(n+\sigma)!}\, T_n^{(\sigma)}(\alpha)\, T_n^{(\sigma)}(\theta)\cos\sigma(\phi - k\beta) + \ldots$$

[where $T_n^{(\sigma)}(\theta) = (1-\mu^2)^{\frac{1}{2}\sigma}\dfrac{d^\sigma P_n}{d\mu^\sigma}$, $\mu = \cos\theta$], and we have to add up $2m$ expressions of this form with the sign changed when κ is odd.

If

$$S_1 = \cos\sigma\phi + \cos\sigma(\phi - 2\beta) + \ldots + \cos\sigma\{\phi - (2m-1)\beta\},$$
$$S_2 = \cos\sigma(\phi - \beta) + \cos\sigma(\phi - 3\beta) + \ldots + \cos\sigma\{\phi - (2m-1)\beta\},$$

we easily see by geometrical considerations or by summing the series, that both S_1 and S_2 vanish unless σ is a multiple of m; and, further, that their difference will also vanish unless σ be an odd multiple of m, in which case it is equal to $2m\cos\sigma\phi$.

Hence we need only consider values of σ of the form $(2p-1)\,m$.

Now the integral

$$\int_0^\pi T_n^{(\sigma)}(\alpha)\sin\alpha\, d\alpha \left(= \int_{-1}^{1} T_n^{(\sigma)}\, d\mu\right)$$

evidently vanishes if $n+\sigma$ be odd. Hence if m be odd we need consider only odd values of n and if m be even only even values of n.

If we exclude for the present the case $m=2$ the term involving Q_2 will disappear in the summation (the first two terms not involving Q plainly disappear in all cases), and putting $\int_{-1}^{1} T_n^{(\sigma)}\, d\mu = A_{n\sigma}$, we have for the velocity potential V

$$V = -\frac{2m\omega}{\pi} r^2 \Sigma\Sigma \frac{2n+1}{(n-2)(n+3)} A_{n\sigma}\frac{(n-\sigma)!}{(n+\sigma)!} T_n^{(\sigma)}(\theta)\cos\sigma\phi$$
$$+ \frac{4m\omega}{\pi}\Sigma\Sigma \frac{2n+1}{n(n-2)(n+3)}\frac{r^n}{a^{n-1}} A_{n\sigma}\frac{(n-\sigma)!}{(n+\sigma)!} T_n^{(\sigma)}(\theta)\cos\sigma\phi,$$

where σ takes the values m, $3m$, $5m$, ..., those values only being considered which are not greater than n.

We can effect the summation in the coefficient of r^n in this expression for V as follows. We have, if ϕ lies between 0 and β $\left(\text{excluding case } \beta = \dfrac{\pi}{2}\right)$ since $\beta = \dfrac{\pi}{m}$,

$$\frac{\sin(2\phi - \beta)}{\cos\beta} = -\frac{8}{\beta}\left\{\frac{\cos m\phi}{m^2-4} + \frac{\cos 3m\phi}{9m^2-4} + \ldots + \frac{\cos\sigma\phi}{\sigma^2-4} + \ldots\right\},$$

σ being of the form $2p-1$.

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Again $T_n^{(\sigma)}$ satisfies the equation

$$\frac{d}{d\mu}\left\{(1-\mu^2)\frac{dT_n^{(\sigma)}}{d\mu}\right\}+n(n+1)\,T_n^{(\sigma)}=\frac{\sigma^2}{1-\mu^2}T_n^{(\sigma)},$$

whence we easily obtain

$$\int_{-1}^{1}(1-\mu^2)\,T_n^{(\sigma)}\,d\mu=\frac{\sigma^2-4}{(n-2)(n+3)}\int_{-1}^{1}T_n^{(\sigma)}\,d\mu=\frac{\sigma^2-4}{(n-2)(n+3)}A_{n\sigma}.$$

Let us now expand $\sin^2\theta\,\dfrac{\sin(2\phi-\beta)}{\cos\beta}$ in a series of tesseral harmonics, thus

$$(1-\mu^2)\frac{\sin(2\phi-\beta)}{\cos\beta}=\Sigma\Sigma C_{n\sigma}T_n^{(\sigma)}(\theta)\cos\sigma\phi,$$

then multiplying both sides of this equation by $T_n^{(\sigma)}(\theta)\cos\sigma\phi$ and integrating over the lune of the sector to determine $C_{n\sigma}$, we get

$$-\frac{8}{\beta}\frac{1}{\sigma^2-4}\frac{\sigma^2-4}{(n-2)(n+3)}A_{n\sigma}=C_{n\sigma}\int_{-1}^{1}\{T_n^{\sigma}\}^2d\mu=C_{n\sigma}\frac{(n+\sigma)!}{(n-\sigma)!}\frac{2}{2n+1},$$

therefore $\quad C_{n\sigma}=-\dfrac{4}{\beta}\dfrac{2n+1}{(n-2)(n-3)}\dfrac{(n-\sigma)!}{(n+\sigma)!}A_{n\sigma}.$

Comparing this result with the coefficient of r^2 in the expression for V, we see that this coefficient becomes

$$\frac{2m}{\pi}\frac{\beta}{4}\Sigma\Sigma C_{n\sigma}T_n^{(\sigma)}(\theta)\cos\sigma\phi$$

$$=\tfrac{1}{2}\sin^2\theta\frac{\sin(2\phi-\beta)}{\cos\beta}.$$

Hence

$$V=\tfrac{1}{2}\omega r^2\sin^2\theta\sin(2\phi-\beta)\sec\beta$$

$$+\frac{4\omega}{\beta}\Sigma\Sigma\frac{2n+1}{n(n-2)(n+3)}\frac{r^n}{a^{n-2}}\frac{(n-\sigma)!}{(n+\sigma)!}A_{n\sigma}T_n^{(\sigma)}(\theta)\cos\sigma\phi.$$

[It is easy to verify from this expression that

$$\frac{dV}{r\sin\theta\,d\phi}=\omega r\sin\theta,$$

when $\phi=0$ and $\phi=\beta$, and that $\dfrac{dV}{dr}=0$ when $r=a$].

If now T denote the kinetic energy of the liquid, we have $2T=-\rho\int V\dfrac{dV}{dn}dS$, and since both V and $\dfrac{dV}{dn}$ have the

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opposite sign over the boundary $\phi = \beta$ to what they have over the boundary $\phi = 0$, it is sufficient to take the integral over this latter and double it.

Therefore

$$2T = -2\rho \int_0^{\pi} \omega \sin\theta \, d\theta \int_0^{a} Vr^3 dr \, d\theta,$$

since

$$\frac{dV}{dn} = \omega r \sin\theta.$$

Effecting the integration we obtain

$$2T = \tfrac{4}{15}\omega^2 a^5 \tan\beta - \frac{8\omega^2 a^5}{\beta} \Sigma\Sigma \frac{2n+1}{n(n-2)(n+3)} \frac{(n-\sigma)!}{(n-\sigma)!} [A_{n\sigma}]^2.$$

It remains to determine $A_{n\sigma}\left(= \int_{-1}^{1} T_n^{(\sigma)} \, d\mu\right)$ as follows.

It may easily be shewn that

$$T_n^{(\sigma)} + (n+\sigma-1)(n-\sigma+2) \, T_n^{(\sigma+2)} = 2(\sigma-1) \frac{\mu}{\sqrt{(1-\mu^2)}} T_n^{(\sigma-1)}.$$

Whence, integrating the right-hand side by parts, we obtain without difficulty

$$\int_{-1}^{1} T_n^{(\sigma)} d\mu = \frac{(n+\sigma-1)(n-\sigma+2)}{\sigma-2} \int_{-1}^{1} T_n^{(\sigma-2)} d\mu.$$

By repeated applications of this result we reduce the integral to $\int_{-1}^{1} T_n^{(1)} d\mu$ when n and σ are odd, and to $\int_{-1}^{1} T_n^{(2)} d\mu$ when n and σ are even.

Now

$$\int_{-1}^{1} T_n^{(1)} d\mu = \int_{-1}^{1} \sqrt{(1-\mu^2)} \frac{dP_n}{d\mu} d\mu = \int_{-1}^{1} \frac{\mu}{\sqrt{(1-\mu^2)}} P_n d\mu$$

$$= \pi \frac{n}{n+1} \left\{\frac{1.3 \ldots n-2}{2.4 \ldots n-1}\right\}^2,$$

Todhunter's *Laplace's Functions*, p. 115.

And

$$\int_{-1}^{1} T_n^{(2)} d\mu = \int_{-1}^{1} (1-\mu^2) \frac{d^2 P_n}{d\mu^2} d\mu = \int_{-1}^{1} 2\mu \frac{dP_n}{d\mu}$$

$$= 2 \int_{-1}^{1} \left(\frac{dP_{n-1}}{d\mu} + nP_n\right) d\mu = 4,$$

since n is even and $[P_{n-1}]_{-1}^{1} = 2$.

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We obtain finally

$$A_{n\sigma}=\int_{-1}^{1} T_n^{(\sigma)} d\mu = \frac{2\pi}{n+1}\,\frac{1}{2^{2m}}\,\frac{(n-1)!}{\{\frac{1}{2}(n-1)!\}^2}\,\sigma\,\frac{(n+\sigma)!}{\frac{1}{2}(n+\sigma)!\,\frac{1}{2}(n-\sigma)!},$$

when n and σ are odd; and

$$A_{n\sigma}=\frac{\frac{1}{2}n!\,\frac{1}{2}(n-2)!}{(n+1)!}\,\sigma\,\frac{(n+\sigma)!}{\frac{1}{2}(n+\sigma)!\,\frac{1}{2}(n-\sigma)!},$$

when n and σ are even.

As a rough verification for the value of $2T$ let us diminish β indefinitely. We easily see that $2T$ reduces to its first term, that is, $T=\frac{2}{5}\omega^2 a^5\beta$ ultimately. But this is the expression for the kinetic energy of a solid vector revolving with the same angular velocity, as it evidently should be, since the liquid in the indefinitely thin wedge behaves practically as if it were solidified.

I have not succeeded in reducing the expression for $2T$ by carrying out the summation with regard to σ, except in the case of the *hemisphere* ($m=1$), for which it will be found by taking Oy as the axis from which θ is measured, and measuring ϕ from Ox, that all the terms, except those for which $\sigma=1$, vanish, and

$$2T=-2\omega^2\pi\rho a^5\,\Sigma\,\frac{2n+1}{n(n-2)(n+3)^2}\,\frac{(n+1)!\,(n-1)!}{2^{2n}\{\frac{1}{2}(n+1)!\,\frac{1}{2}(n-1)!\}^2},$$

the summation extending to all odd values of n.

The case (omitted above) of $m=2$, or the quadrantal sector, presents no difficulty, the only difference being that we have a term of the form $Ar^2\left(\frac{1}{2}+\log\frac{a}{r}\right)\sin^2\theta\cos 2\phi$ in V, and that we cannot effect the summation in the coefficient of r^2 as before, but must in finding $2T$ integrate the terms in it and add them on to the second $\Sigma\Sigma$.

NOTE.—If we calculate the value of $2T$ for a revolving hemispherical shell from the above formula, we obtain, taking the first few terms, and assuming the remainder approximately,

$$2T=\cdot 1592\pi\rho\omega^2 a^5.$$

Now it is easily proved that, in the case of a cylinder whose base is a semi-circle, the value of $2T$ per unit length of the cylinder is

$$\pi\rho\omega^2 c^4\,\frac{16-\pi^2}{4\pi^2},$$

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the axis of rotation passing through the centre of the completed circle, and c being the radius of the circle.

Now suppose the liquid in our hemisphere constrained by an indefinite number of fixed rigid planes all perpendicular to the axis of rotation. Then the motion is reduced to two dimensions, and evidently

$$2T = 2\,\frac{16-\pi^2}{4\pi^2}\,\pi\rho\omega^2\int_0^a (a^2-h^2)^2\,dh$$

$$= \tfrac{4}{15}\,\frac{16-\pi^2}{\pi^2}\,\pi\rho\omega^2 a^5$$

$$= \cdot 1656\pi\rho\omega^2 a^5,$$

which is, as it should be, greater than the value of $2T$ for the unconstrained liquid.

NOTE ON TAYLOR'S THEOREM.

By *Rev. Henry George Day, M.A.*, St. John's College, Cambridge.

THE object of this theorem is to exhibit, in an ascending series of powers of h, the form of

$$f(a+h)-f(a),$$

that is of

$$\int_0^h f_1(a+x)\,dx,$$

where $f_r(a+x)$ is the r^{th} differential coefficient, according to x, of

$$f(a+x).$$

Integrating by parts, in the ordinary manner,

since $\int f_1(a+x)\,dx = -(h-x)f_1(a+x) + \int (h-x)f_2(a+x)\,dx,$

we get $$\int_0^h f_1(a+x)\,dx = hf_1(a) + \int_0^h (h-x)f_2(a+x)\,dx,$$

and similarly

$$\int_0^h (h-x)f_2(a+x)\,dx = \frac{h^2}{2!}f_2(a) + \int_0^h \frac{(h-x)^2}{2!}f_3(a-x)\,dx.$$

Continuing this process, if the series be convergent, we obtain

$$f(a+h)-f(a) = hf_1(a) + \frac{h^2}{2!}f_2(a) + \frac{h^3}{3!}f_3(a) + \frac{h^r}{r!}f_r(a) + \ldots ad\ inf.,$$

which is Taylor's Theorem.

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Again, the remainder after the term in h^r is

$$\int_0^h \frac{(h-x)^r}{r!} f_{r+1}(a+x)\,dx,$$

which evidently lies between $A\dfrac{h^{r+1}}{(r+1)!}$ and $B\dfrac{h^{r+1}}{(r+1)!}$, where A and B are the greatest and least values of $f_{r+1}(a+x)$, as x varies from 0 to h.

This is Laplace's theorem of the limits of the remainders.

SOME ELLIPTIC FUNCTION FORMULÆ.

By *Rev. M. M. U. Wilkinson.*

IF the modulus $k^2 = -\omega$, where $\omega^2 + \omega + 1 = 0$, we have

$$\operatorname{sn}(\alpha+\omega\beta+\omega^2\gamma) = 1 - \operatorname{cn}(\beta-\gamma)\operatorname{cn}(\gamma-\alpha)\operatorname{cn}(\alpha-\beta), \div$$

$$\operatorname{cn}(\alpha+\omega\beta+\omega^2\gamma) = \omega^2\operatorname{sn}(\beta-\gamma)\operatorname{dn}(\alpha-\beta) - \operatorname{sn}(\gamma-\alpha)\operatorname{cn}(\alpha-\beta), \div$$

$$\operatorname{dn}(\alpha+\omega\beta+\omega^2\gamma) = \operatorname{sn}(\alpha-\beta)\operatorname{dn}(\gamma-\alpha) - \omega\operatorname{sn}(\beta-\gamma)\operatorname{cn}(\gamma-\alpha), \div$$

$$\text{denom.} = \omega\operatorname{sn}(\gamma-\alpha)\operatorname{dn}(\beta-\gamma) - \omega^2\operatorname{sn}(\alpha-\beta)\operatorname{cn}(\beta-\gamma).$$

The formulæ can be easily established from the identities

$$\omega\operatorname{sn}x = \operatorname{cn}x\operatorname{sn}\omega x, \quad \omega^2\operatorname{sn}x = \operatorname{dn}x\operatorname{sn}\omega^2 x,$$

$$\operatorname{dn}x = \operatorname{cn}x\operatorname{cn}\omega x, \quad 1 = \operatorname{dn}x\operatorname{cn}\omega^2 x,$$

$$1 = \operatorname{cn}x\operatorname{dn}\omega x, \quad \operatorname{cn}x = \operatorname{dn}x\operatorname{dn}\omega^2 x,$$

and the ordinary addition-formulæ.

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